In E. A. Silver & P. A. Kenney (Eds.), 2016. More lessons learned from research Volume 2: Helping all students understand important mathematics. (pp. 79-91). Reston, VA: NCTM.

Chapter 7

Class Learning Zone and Class Learning Paths

Responsive Teaching in First-Grade Mathematics

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In the Japanese grade 1 classroom, students are busily figuring out how to add two numbers, 9 and 4. The teacher, Mr. Otani, writes the number sentence 9 + 4 on the board and puts 9 blue and 4 red magnets in a row on the board. Some students share different counting strategies (count all, count on, count by 2s, etc.). Others share how to make a ten (9 + 4 = 9 + 1 + 3 = 10 + 3 = 13). Koichi goes up to the board and rearranges the magnets to show his thinking (see fig.7.1).

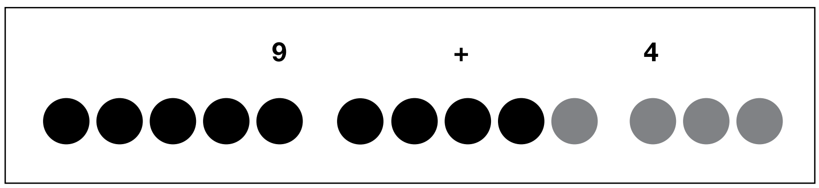

Fig. 7.1. Koichi's method of 9 + 4 = 5 + 5 + 3

Koichi: I made groups of 5 and 5.

Mr. O: I wonder what is different about this method . . .

Students: He did 5 + 5 + 3!

Mr. O: Yes. How did you think of this, Koichi?

Koichi: I thought this way: 5 and 5 is 10, so 3 more is 13. Is it OK?

Students: It is OK! *[shouting together]*

Through the case study of a Japanese classroom, this chapter will illustrate how facilitation of different levels of understanding and fluency can support all students in learning mathematics. We will discuss how different instructional supports can be coordinated for teachers to make such classroom teaching and learning possible.

National reports summarizing research describe a new view of teaching mathematics. This approach builds competence in culturally valued knowledge by relating such knowledge to what students already know and balances conceptual understanding and procedural fluency to develop mathematical proficiency. This chapter presents the ZPD Mathematical Proficiency Model to support the implementation of these views. Based on the original theory of Vygotsky (1978, 1986), Tharp and Gallimore (1988) described teaching that occurs when assistance is offered at points in the Zone of Proximal Development (ZPD) when performance

This chapter is adapted from A. Murata & K. Fuson (2006), Teaching as assisting individual constructive paths within an independent class learning zone: Japanese first graders learn to add using 10, *Journal for Research in Mathematics Education*, *37*, 421–456.

requires assistance. The ZPD is defined as the distance between the child's actual developmental level and his or her potential development under the guidance of or in collaboration with a more experienced partner. Thus, teaching is a shared social activity. In our model, assistance is offered only when needed; decreasing amounts of assistance are needed as students progress through their own learning paths in any given topic, creating mutual understanding between the teacher and the learner for giving assistance adapted to the learner. This model underscores the view of learning as a constructive activity by the learner so that the internalization process does not involve rote copying of behavior.

Such teaching is well documented in many cultures and in many different activity settings around the world. However, it often occurs with one learner and one teacher. So how can such an ideal view of teaching possibly work in a classroom with one teacher and as many as twenty or even thirty-five students? We propose a perspective to illustrate how our definition of teaching could be enacted for individual students within the whole-class setting: *Class Learning Zone with a Class Learning Path.* Key to this perspective is our knowledge from research that, for many mathematics topics, a few typical errors stem from partial and incomplete understandings, and some other, more random errors arise from momentary lapses of attention or effort. Likewise, there are usually several solution methods, but these are limited in number and vary in their sophistication, generalizability, and ease of understanding. Thus, for any given mathematics topic, there are not twenty or thirty-five different learning paths or strategies for the teacher to understand and assist. Instead, there are usually three to six strategies that may have minor variations, and these can be noted in curricular materials that assist teachers in learning to assist students. Also, visual supports can be developed and shared with teachers to aid them in teaching particular topics. Of course for any mathematics topic or problem, there is always a possibility of new solutions or strategies, and not all can be anticipated, so the class Learning Path needs to be responsive to such possibilities.

When teaching with the idea of a Class Learning Zone, a teacher would orient students to the new instructional topic and then elicit from students their methods for solving such problems or for thinking about such contexts. With assistance from teaching materials, the teacher begins moving along a Class Learning Path that will provide assistance to move students forward to a good-enough and culturally valued general solution, with individual students starting from their own initial knowledge. The Class Learning Zone is the day-to-day learning zone within which the teacher organizes assistance for various students. Exceptional students (either extremely advanced or extremely delayed) may fall outside the Class Learning Zone. The former may help others but may need assistance to do so (or to want to do so). The Class Learning Path is the day-to-day sum of the learning paths of most of the students in the class (reflecting the state of growth in their methods and in their understanding each day), but this falls within manageable groups of related-enough mathematical assistance needs. A few students may not fully master a target solution method, but assistance will continue in subsequent units toward mastery or with additional help outside of class. Students may also continue to use any powerful or general-enough method of their own choice. Learning for all students includes increased understanding of how other students solve problems and increased ability to assist other students.

Figure 7.2 illustrates the ZPD Mathematical Proficiency Model. There are four stages for a learner to move through the ZPD to achieve a given performance goal: Phase 1 is assistance provided by more-capable others; Phase 2 is assistance provided by the self (as the means of assistance of others are internalized into speech-for-self); Phase 3 is internalization-automatization-fossilization; and Phase 4 is de-automatization with recursion through the stages as performance that was once mastered slips away over time. Decreasing assistance over time is part of responsive assistance.

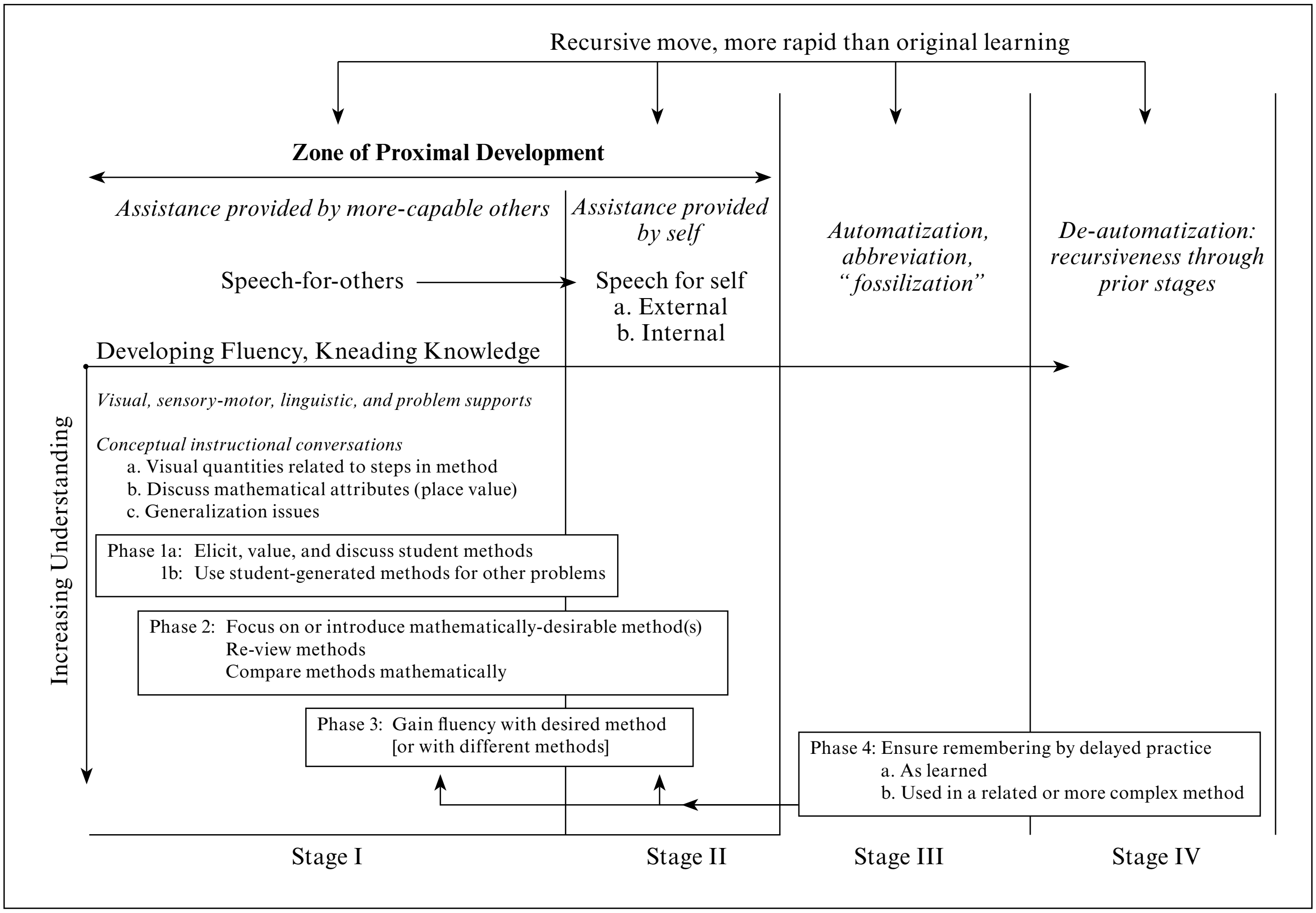

Figure 7.2. Stages of Learning and Class Learning Zone Phases in the ZPD Mathematical Proficiency Model

We identified in the ZPD model two independent but continually interacting aspects of teaching over time: developing understanding and developing fluency. Fluency moves to the right horizontally through the four stages. Understanding moves down vertically and is central in the four phases. Visual, sensory-motor, linguistic, and problem supports provide the bases for building understandings by all within conceptual instructional discussions that assist such understandings. These supports are conceptual tools that form the backdrop for all of the collective and individual functioning within the Class Learning Zone. Their use can continue at any stage or phase of learning in the ZPD. The kinds of learning supports (conceptual tools) and the mathematical points within the conceptual discussion vary with the mathematical topic. The necessity of identifying the learning supports for particular topics is part of our model.

In the following sections, we illustrate this model in action using a Japanese grade 1 classroom example. We offer an overview of the assistance that one Japanese teacher provided as students learned a culturally valued mathematics concept, how he changed the levels of support for the class and for individual students as the instructional unit progressed, and how he used teaching supports and tools. We also see how students assisted other students. This model and the teaching we see are consistent with the Common Core State Standards

(National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) and with its Standards for Mathematical Practices. The method discussed here is in a grade 1 standard (1.OA.C.6), and all eight mathematical practices are exemplified in our discussions of teaching and learning.

The goal of the unit chosen for our case study was to learn to add numbers with totals in the teens. This unit was chosen because it involves learning a complex multistep method, the Break-Apart-to-Make-Ten (BAMT) method, which is specified in the Japanese National Course of Study. The complexity of this method pushed our concept of a Class Learning Zone because this method is demanding for students. Student understanding of and fluency with the BAMT method are viewed as important for their future learning of multidigit addition and subtraction in the curriculum because it helps them make sense of and use the values of 10-ness in the number system, and it is a general addition method useful in multidigit addition, where students will be moving the new group of 10 to the next left column. This method also prepares students for related methods for subtraction.

Method

The data were collected in a grade 1 classroom at a full-day Japanese school in a suburb of a Midwestern metropolitan city in the United States. The school is operated by the Japanese Ministry of Education and closely follows the Japanese National Course of Study. Administrators and teachers are sent directly from Japan through the ministry, and the instructional language is Japanese. The school primarily serves Japanese families who are in the area for a short period of time (two to five years), and the community puts much effort into preserving Japanese culture in their lives as well as maintaining Japanese ways of teaching and learning. The grade 1 teacher, Mr. Otani, had taught in Japan prior to coming to the school. Twenty-five students were in the classroom.

Data were collected over eleven lessons in a three-week period during the fifth month of the school year. For each observation, lessons were videotaped and careful field notes were taken. The methods the students used in the classroom as they solved addition problems were also noted. Six target students (two with higher, two with medium, and two with lower performance levels, as identified by the teacher) were interviewed as they solved problems before and after the instructional unit and at the end of the school year.

Data from the observation field notes were analyzed to illustrate how Mr. Otani (1) provided assistance as students learned the steps of the BAMT method and (2) changed his support levels for individual students and also over time. Figure 7.3 shows the steps that students used to carry out this method for 9 + 4 as well as a representational drawing taken from the Japanese teachers' manual (Tokyo Publishing, 2000) and used in the classroom. Figure 7.4 shows the levels of support Mr. Otani provided for students across lessons.

Field notes were coded for the external problem-solving steps that students took in the whole-class context and individually in independent work as they learned, and for the kinds of support Mr. Otani provided for the steps. Other means of support were also identified. Videotaped data were reviewed to verify the data coded from the field notes. For individual student learning, the different methods and the steps of the BAMT method used by the six target students were also analyzed.

Teaching Phases and Teacher Support

In the following sections, we describe how Mr. Otani supported student learning through the four phases of the instructional model.

Steps of the BAMT method	Step 1 Find that 9 needs 1 more to make 10	Step 2 Separate 4 into 1 and the rest (3)	Step 3 Add 9 and 1 to make 10	Step 4 Add 10 and 3 to make 13
Counter use makes objects change from 9 + 4 to 10 + 3	Count 9 counters and 4 counters, then move 1 from 4 to make a group of 10 with 9.	See 3 left in 4.	Just see and think 9 and 1 is 10 (making of 10 already happened in step 1).	See 10 counters and see 3 counters and think ten-three or count on "ten-one, ten-two, ten-three."
Finger use makes each step visible separately	Open 9 fingers, see 1 more finger is folded to reach 10.	Open 4 fingers, fold 1, and see 3 fingers are still left.	Open 9 fingers, open 1 more, and see 10 fingers, or remember it has been done already with step 1.	Open 10 fingers, say "ten," fold them again, and open 3 more fingers and count-on as they are folded, ["ten-one, ten-two, ten-three."] or know 10 and 3, 13.
Visual representational drawings, make a numerical trace of old and new problems and of steps in the change process visible	9 + 4 / 1 The line under 4 toward the place between 9 and 4 helps students know they need to think of 9's partner to make 10.	9 + 4 / \\ 1 3 Two lines under 4 indicate how the number 4 is separated into two partners.	9 + 4 10 / \\ 1 3 Circling of 9 and 1 shows how two numbers are combined to make 10.	9 + 4 = 13 10 / \\ 1 3 Shows 3 is the only number that is not yet a part of 10. So 10 + 3.

Note: The first two steps are also facilitated by the linguistic support of the term "partners" for the two addends that form the totals. The final step is also facilitated by the Japanese linguistic form of 13 as "ten three."

Fig. 7.3. Teaching supports and their facilitation of the BAMT steps (example: 9 + 4)

Phase 1: Elicit, value, and discuss student methods (lessons 1 and 2)

For the initial introduction of the unit, Mr. Otani showed a group of 9 blue and 4 red magnetic counters in a row on the board. Some students immediately shouted out the answer, "13!" Mr. Otani then initiated discussion by saying, "Some of you are quick in telling the answer, but who can share with the class your thinking?" Several students raised hands to share their ideas. Sakiko went to the board when called and moved 1 red counter to add to the blues.

Sakiko*:* From 4, I add 1 to 9 to make 10.

Mr. O: So, the 9 became 10 and 4 became 3?

Sakiko: Then we know 10 and 3 make 13. We learned that before.

Mr. Otani summarized Sakiko's method on the board and continued to ask for other students' contributions to drive the discussion.

Mr. O: Did anyone do this differently? 9 + 4? Nobuhiko?

Nobuhiko: 9 + 4 is . . . at first, 3 and 4 is 7.

Students: What? What are you saying? We don't understand!

Mr. O: Will you say it again, Nobuhiko?

Nobuhiko: I took 3 from 9 . . . *[moves 3 counters from 9]*

Goals	Make 9 into 10 with part of the other addend	Find how many more are left to add to 10	Make 10 for new problem, 10 + 3,	In new problem. 10 + 3, find total
Steps	Step 1 Find that 9 needs 1 more to make 10	Step 2 Separate 4 into 1 and the rest (3)	Step 3 Add 9 and 1 to make 10	Step 4 Add 10 and 3 to make 13
Visual representational support in textbooks and on the board	9 + 4 / 1	9 + 4 / \\ 1 3	9 + 4 10 / \\ 1 3	9 + 4 =13 10 / \\ 1 3
Level A Support: Steps 1, 2, 3, 4	"9 and what number make 10?" (Teacher points to 9.)	Teacher draws sticks to elicit break-apart partners for 4. "What two numbers are you separating 4 into (to make 10)?"	"What do 9 and 1 make?" (Teacher circles 9 and 1, writes 10 next to the circle.)	"What do 10 and 3 make?" (Teacher points to numbers 10 and 3, says "ten and three" to make connections to the total "ten-three.")
Level B Support: Steps 2, 3, 4		Teacher draws sticks to elicit break-apart partners for 4. "What two numbers are you separating 4 into (to make 10)?"	"9 and 1 make . . . ?" (Teacher points to 9 and 1).	"10 and 3 make . . . ?" (Teacher points to 3.)
Level C Support: Steps 2 and 4		Teacher draws sticks to elicit break-apart partners for 4. "4 is what number and what number?"		"10 and 3 make . . . ?" (Teacher points to 3.)
Level D Support: Step 2 visually		Teacher draws sticks to elicit break-apart partners for 4. No verbal guiding.		
Level E Support: Step 4 visually and with partners		Break-apart partners are filled in from level D. No verbal guiding.		(no guiding question)

Notes: Each level supports fewer steps. Levels D and E often occurred in combination to support the learning process. For level D, Mr. Otani all but once elicited only step 2 and students typically gave break-apart partners of the addends. Following this step, level E support occurred when the break-apart partners remained on the board for a visual cue while students stated answers to problems without verbal guiding.

Fig. 7.4. Steps, drawing, and levels of teacher assistance for learning the BAMT method

Mr. O: You took 3 from 9?

Nobuhiko: Add that 3 and 4 . . . *[puts 3 and 4 counters together]*

Mr. O: So, 9 is . . .

Nobuhiko: 9 became 6. Separate 9 into 3 and 6 . . . Then, 7 . . . I mean . . . Because 7. 4 and 3 became 7, so 7 and 6 is 13. *[points to the counters]*

Mr. O: 13. 7 and 6 make 13. You like 7 + 6, don't you?

Students: Nobuhiko has remembered that way for a long time!

Mr. O: Is that so?

Students: Yeah, he always does that way. This is Nobuhiko's secret method!

Students worked to understand Nobuhiko's solution method. His idea, which seemed incomprehensible at the beginning ("at first 3 and 4 is 7"), was gradually clarified with guiding questions by Mr. Otani. As other students made sense of his thinking, they accepted his method as a valid mathematical approach. As Nobuhiko worked to articulate his idea to his classmates, he explained one step at a time. This careful reflection on his own thinking helped develop mutual understanding in the community. Other students actively evaluated the quality of Nobuhiko's explanation by demanding that he clearly describe the process of his thinking. In these ways, Mr. Otani and his students negotiated and established shared understanding of what a good mathematical solution and explanation of such a solution should be. The fact that this method is an atypical method (it was not listed in the Teachers' Manual) emphasizes how a Class Learning Zone can involve idiosyncratic student thinking and not just typical methods.

In this first lesson of the unit, Mr. Otani encouraged students to share their ideas based on their spontaneous thinking and prior knowledge and allowed room for diverse methods, including counting all (see our original 2006 article for all methods shared). The sharing process at the beginning of the unit provided opportunities for students to review previously learned concepts, demonstrate their competence, and set the stage for future exploration. Mr. Otani carefully directed student discussion to focus on the process of solving the problem, thus providing opportunities both for the students who already knew the answer and for those who were experiencing such a complex problem for the first time.

Phase 2: Focus on the BAMT method (lessons 2 to 4)

Figure 7.5 shows the changing levels of support provided by Mr. Otani related to changing lesson activities in the instructional unit. In the unit, the problems were introduced by the same first addend, $9 + n$, as it is easier for students to see how 9 needs 1 more to make 10 and to decompose the second addend into $1 + n$ (step 2 of the BAMT method). By the middle of the unit, students re-viewed[1] the ten partners of 9, 8, and 7 as a way to explore $6 + n$ problems. These re-views not only helped refresh students' memories but also connected their previous knowledge to the new topic. When a new first number was introduced, it was typically introduced as an extension of solving the problem with a more familiar number. For example, in introducing, $7 + n$ addition problems in lesson 5, Mr. Otani first supported students with $9 + n$ and $8 + n$ problems on the board by asking individual students to solve them. He then wrote $7 + n$ problems on a small portable board, placed it next to the $8 + n$ problems, and continued the previous questioning pattern. Placing different types of problems side-by-side highlighted the similarities between the problem sets. From day 2 to day 4, Mr. Otani shifted the conceptual emphasis of the re-views from the first step to the general pattern in the second step for all problems beginning with 9 to a short-cut way of thinking about these problems.

Mr. Otani also led the students to compare different methods shared in the discussion and had them vote on what they each considered to be the most accessible method. While the votes split at the beginning of the unit, over time the students came to see the value of the BAMT method.

[1] We write "re-view" with a hyphen to convey the substance of these sessions in Japanese schools. The "pre-view and re-view" learning routine is embedded in daily practices.

	Activities	Level of Support Used						
		A	B	C	D	E	No	V
1	1. Whole-class exploration of different methods for 9 + 4							
	2. Voting for the easiest method [IC]							
2	1. Whole-class re-view of methods (9 + 4) [IC]							
	2. Voting for the easiest method [IC]							
	3. Discussion of place-value and the BAMT method [IC]							
	4. Whole-class intro for 9 + #							
	a. Step 1 for the set of 6 problems (discussion of 9's partner to make 10 [IC])	As						
	b. Step 2 for the set of 6 problems	As						
	c. Step 3 for the set of 6 problems	As						
	d. Step 4 for the set of 6 problems	As						
	5. Voting for the easiest method [IC]							
3	1. Whole-class re-view of methods (9 + 4) [IC]							
	2. Whole class practice of 9 + # (3 problems) steps 1, 2, 3, 4	Ap						
	3. Individual practice, 9 + # (4 problems)							
	4. Individual-in-whole-class practice of 9 + # (problems from 3)							V
	a. Step 1 for the set of 4 problems	As						
	b. Step 2 for the set of 4 problems	As						
	c. Step 3 for the set of 4 problems	As						
	d. Step 4 for the set of 4 problems	As						
	5. Discussion of 9's partner to make 10 [IC]							
	6. Voting for the easiest method [IC]							
4	1. Whole-class re-view of the BAMT method, 9 + 3, steps 1, 2, 3, 4 [IC]	Ap						
	2. Whole-class re-view of 9 + # (6 problems) steps 2, 4 [IC]							
	3. Individuals-in-whole-class review of 9 + # (problems from 2) steps 2 and 4			Cp Cp				
	4. Individuals-in-whole-class practice of 9 + # (problems from 2), BA partners written on the board (other things erased), Ss say answers, 6 problems					Ep		
	5. Individual-in-whole-class practice of 9 + # (problems from 2), BA erased, Ss say answers				Dp			
	6. Whole-class intro for 8 + # (8 + 3), steps 2, 3, 4 [IC]							
	7. Individuals-in-whole-class practice, 8 + # (7 problems)		Bp					
	a. Step 2 only, with break-apart sticks							
	b. BA written on the board, Ss say answers, T points to random problems				Ds	Ep		
5	1. Whole-class re-view of 9 + 5 and 8 + 6, steps 2, 3, 4 [IC]		Bp					
	2. Individual-in-whole-class practice of 15 mixed 9 + #, 8 + #							
	a. Step 2 only, with break-apart sticks				Ds			
	b. BA written on the board (other things erased), Ss say answers, T points to random problem					Ep		
	3. Individual-in-whole-class intro of 7 + # (6 problems) [IC]							
	a. Step 2 only, with break-apart sticks				Ds			
	b. BA written on the board (other things erased), Ss say answers					Ep		
	4. Whole-class say answers to 7 + #, with BA partners written							
	5. Individual practice of 7 + # (4 problems)					Ep		V

Fig. 7.5. Levels of support over eleven lessons

	Activities	Level of Support Used						
		A	B	C	D	E	No	V
6	1. Whole-class intro of 6 + 5 by discussing 10 partner for 6, re-views of 10 partners for 9, 8, 7 [IC]							
	2. Individual practice of 6 + # (5 problems)							V
	3. Individual-in-whole-class report of 6 + # (problems from 2) while the rest of the class gave feedback, "It is OK!"						No	
	4. Individual practice for 16 mixed 6 + #, 7 + #, 8 + #, and 9 + #							V
7	1. Individual-in-whole-class report answers on problems solved in lesson 6; T writes equation and answer as it is shared, class gives feedback, "It is OK!"						No	
	2. Whole-class intro for smaller + larger (4 + 8, equation and answer only) [IC]							
	3. Individual practice of 12 smaller + larger; teacher notices that many students are counting on, so shifts to 4.							V
	4. Whole-class discussion on smaller + larger, 2 + 9, steps 1, 2, 3, 4, solved from 9 and from 2 [IC]	Ap						
8	1. Individual practice of 11 smaller + larger problems							V
	2. Individual-in-whole-class report answers on problems just solved (as in lesson 7, 1 above)						No	
	3. Individual practice of 2 word problems							V
	4. Individual-in-whole-class report on problems just solved (disagreement on quantifiers)						No	
9	1. Individual practice on 8 mixed problems							V
	2. Individual-in-whole-class report answers on problems just solved (as in lesson 7, 1 above)						No	
	3. Individual practice on 8 mixed problems							V
10	1. Like lesson 9 (no observation)							V
11	1. Whole-class report answers on 8 mixed problems solved in previous class (as in lesson 7, 1 above)						No	
	2. Individual practice on 6 mixed problems							V
	3. Individual-in-whole-class report answers on problems just solved (as in lesson 7, 1 above)						No	

Fig. 7.5. ***Continued***

Notes: The support always involved the drawing on the board and sometimes (especially for individuals) also involved fingers or counters. The support identified is standard support for the class. Some individuals might have received more support. Small "s" and "p" placed after the letter that shows the support level mean "steps" and "problems." For example, "As" means level A support for a step; "Ap" means level A support for solving the whole problem. "No" means no support. "V" means varied support with students (for individual practice). [IC] means instructional conversation.

Part 3: Gaining fluency with the BAMT method (lessons 5 to 11)

When the word *practice* is used in Japanese classrooms, it conveys a meaning slightly different from what it does in English. The Japanese word *practice* is written as a combination of two Chinese characters: 練習; the first character means "kneading" and the second character means "learning." Together, the characters represent the meaning of kneading different ideas and experiences together to learn. Such kneading was observed in all three of the different modes of practice identified: whole-class practice, individuals in whole-class practice, and independent practice (see fig. 7.5). Students brought different ideas, experiences, and approaches to learning the BAMT method, and the differences were "kneaded" through various practice forms to support each student's learning as well as to establish a common understanding base in the classroom.

For the whole-class practice, Mr. Otani typically stood in front of the class with a set of problems written on the board. As he asked questions to support students to take a specific step in the BAMT method, he pointed to that part of the problem on the board, and students answered the questions together aloud. Mr. Otani often pointed to the questions on the board in order (e.g., from left to right, from top to bottom), but he sometimes pointed to the questions randomly so students could not think ahead. Sometimes, his questions assisted all of the steps to solve one problem and the same questions were asked to solve the next problem. At other times, he asked questions for one particular step for all the problems on the board, then moved on to the next step for all of the problems. Step support happened in lessons 2 and 3 when students were learning the steps of the BAMT method for the first time; in lessons 4 and 5, this support assisted their learning of step 2 (the most challenging step) for the first addend 8 (lesson 4) and then addend 7 (lesson 5) by combining level D step support and level E problem support. Students were encouraged to speak loudly for all whole-class practices, and they answered with enthusiasm and energy.

Individuals in whole-class practice followed the same pattern as the whole-class practice. With a set of problems written on the board, the same questions were asked, but students took turns answering the questions individually. Students usually answered by their seating order (e.g., starting from the students who sat at the front row of the right side of the room to the students in the back row, then to the students in the next column, etc.). As with the whole-class practice, Mr. Otani changed the order of the problem at times. The most distinctive difference for this individual in whole-class practice is that after one student answered the question, the student always asked the whole class, "Is it OK?" The whole class then answered by shouting together, "It is OK!" if they agreed with the student, or "It is not OK!" if they did not. When they disagreed, Mr. Otani guided the discussion among students to identify and resolve the disagreement.

For independent practice, students worked at their seats solving problems independently. Often, they worked on assigned problems from textbook pages, but as they finished those, they worked on a packet of worksheets Mr. Otani had prepared or a set of calculation cards (small flash cards held together by a ring). When individual students had difficulties, students who sat close by spontaneously helped them.

The interactions among these different modes of practice supported student learning in different ways. The whole-class practice provided a fun and safe group-learning environment where students shouted answers together. Individuals in whole-class practice offered opportunities for individual students to show their developing fluency with the method and get whole-class feedback. Individual practice allowed students to focus on areas where they needed more work and also created a foundation before the whole-class sharing of individual in whole-class reporting answers.

Phase 4: Delayed practice

Delayed practice happened in the re-view section of the textbook where concepts that students learned previously are revisited and practiced. Here, students re-viewed independently in familiar practice contexts. The BAMT method was also used in a related or more complex method in a subsequent unit of subtraction using 10 and in multidigit addition in grade 2.

Shifts in Teacher Levels of Assistance

The steps involved in the BAMT method were not difficult when they were taken one step at a time, because each had been learned in previous units. However, many students experienced difficulty coordinating the steps into a fluent whole. Initially, Mr. Otani supported each step by questions (see level A support in fig. 7.3). He then dropped support one step at a time, eliminating the easier steps 1 and 3 first and keeping support for the most difficult step 2 at the final level (visual only). However, he always increased the levels of support for students who needed it.

Figure 7.5 shows how this full support decreased over time through levels B through E and how it varied with the kinds of problems. On days 4 to 6, assistance decreased as the class continued to practice a given type of problem but increased when they began a new type of problem. From day 4 through day 7, the initial level of assistance at the beginning of the day decreased.

Sometimes more-advanced students spontaneously modeled for the class the BAMT method with fewer steps than the steps Mr. Otani was supporting in the Class Learning Zone. For example, when students in the whole-class practice were experiencing level C support (steps 2 and 4), Sachiko stood up to solve the problem 9 + 5:

Mr. O: *[points to the problem 9 + 5]*

Sachiko: *[starts talking before Mr. Otani can ask guiding questions]* 10 and 4 is 14.

Mr. O: OK, OK, what did you do first?

Sachiko: Separated 5 into 1 and 4, then 10 and 4 is 14. Is it OK?

Students: It is OK!

Here Sachiko followed only step 4, but Mr. Otani elicited from her steps 2 and 4.

On day 7, Mr. Otani introduced a new class of problems in which the smaller number was the first addend (e.g., 2 + 9 instead of 9 + 2). In the individual practice, many students solved these problems by counting on and did not use the BAMT method. When Mr. Otani realized this, he initiated a conversation shifting back to level A support to discuss BAMT solutions for 2 + 9 (making a 10 with the second addend) and related the solutions of 2 + 9 and 9 + 2 to each other. He drew on the board full representational drawings for 2 + 9 (where 2 was broken into 1 and 1 to make a 10 with 9) and then for 9 + 2 (where the same partners of 1 and 1 were shown under 2 but now on the right). He then guided student discussion of these solutions by using two groups of 2 and 9 counters and asking students, "Can we move counters like this and make 10 on this side [for 1 + 1 + 9 = 11]?" and then for the counters 9 and 2, "Can we move counters to make 10 to make 11 this way [for 9 + 1 + 1 = 11]?" He then wrote 2 + 9 and 9 + 2 on top of each other and led a discussion by questioning to help students analyze which of these was easier and to see the similarities between the new situation and the larger-plus-smaller-addend addition situations they had been solving using the BAMT method. Most students quickly went back to use the BAMT method, and most started with the larger number even if it was the second addend.

Mr. Otani's questions also shifted through levels to become more abstract and informal. His questions at the beginning of the unit were explicit directives (e.g., "What number do 1 and 9 make together?"). As the unit progressed, he was more likely to state the same question as a process in action (e.g., "9 and 1 is . . . ?"), or sometimes he only pointed to the numerals on the board as an implied nonverbal question (see fig. 7.4).

Data from individual student interviews and classroom work showed how each target student moved along a unique learning path, which illustrated his or her own learning trajectory. A full discussion of the target students appears in our *JRME* article (Murata & Fuson, 2006).

Table 7.1
Aspects of teaching for understanding and fluency: Examples from the Japanese Class Learning Zone Classroom

Focus on meaning supports (representational and cultural/visual tools) and on conceptual discussion	
Visual, linguistic, and sensori-motor representational support for learning steps	• Use visual representations (physical objects, drawings, and fingers along with oral explanations) to strengthen students' understanding of crucial steps: – Move objects to show 9 becoming 10 – Circle numbers to make 10 – Draw upside-down "v" to show break-apart partners – Emphasize the critical conceptual step by using a colored ten in the drawing • Help students make connections between different representations
Focus on individual mathematical thinking	
Discuss, value, and assist students' ideas and thinking	• Allow students to share ideas and different approaches • Ask questions to guide student thinking • Maintain students' ownership of ideas (call different methods with students' names, vote for different methods after discussing advantages and disadvantages)
Support students' different learning paths	• Vary questioning patterns to meet different levels of understanding of individual students and provide modeling and explanation when needed • Include less-advanced students in whole-class practice to allow them to experience the whole process rapidly, but support their individual solving as necessary with questions and modeling as needed • Consider differences among students as strengths, and create situations where they benefit from the differences
Focus on mathematics	
Support generalization and focus on the mathematics of their learning	• Support generalization of problems with the smaller number first – Introduce and practice problems by their mathematical structure (e.g., 9 + # problems, then 8 + # problems, etc.) to support initial learning – Discuss the similarities and differences of problems according to their mathematical structure (size of first addend) to support generalization • Discuss mathematical aspects of methods (e.g., the new unit of 10 related to place value) – Discuss whether to start with the smaller or larger addend
Focus on assisting all students to speech-for-self, abbreviation, and automatization	
Facilitate fluency	• Provide opportunities to practice with decreasing visual and question support • Pair students and encourage them to practice using flashcards that mix the problem types (practice with immediate feedback) • Send a worksheet packet home that explains to parents what students are learning, and asks them to time the students as they finish their homework. This helps the teacher understand the fluency level of each student.

Teaching for Understanding and Fluency

Our case study gave life to the ZPD Model of Mathematical proficiency. It illustrated how one teacher assisted student learning by valuing students' informal knowledge and approaches, allowing students time and opportunities to explore different ideas, helping bridge the distance between their existing knowledge and the new method, and giving time for students to practice and gain fluency with a newly learned method. Table 7.1 summarizes aspects of the teaching. The focus on meaning supports and on conceptual discussion of individual mathematical thinking, on aspects of the mathematics, and on a learning path helped students develop understanding of the overall method, coordinate and verbalize steps in the multistep method, and develop and move toward fluency. Mr. Otani assisted students of different fluency levels to work together, and he helped individuals to move forward within their own learning path. The visual and verbal question teaching supports were gradually internalized by students as they used them to provide self-assistance to coordinate or carry out particular steps. Mr. Otani assisted community and individual interaction for everyone's learning, including his own, as the class developed shared understanding. The use of consistent visual representational supports kept the community together, as it helped reduce the differences between individual students during whole-class and individual practice.

The students in this grade 1 classroom were always willing and eager to support and adjust their own levels to the levels of their peers whose learning paths were different from their own. The emphasis on relationship and "sameness" in the culture helps create an environment in which students understand difference as a norm but also a changeable characteristic, and thus they try to be like one another. Helping one another is a part of their identities, and that is well supported in various classroom rituals and activities. There are similar examples in U.S. classrooms where teachers work diligently to create a collaborative learning environment to help students learn mathematics. In these classrooms, teachers value collaboration among students over competition, and they create a safe environment in which students are encouraged to share their mistakes, ask questions when they do not understand, and provide explanations when they are not completely sure. Teachers can model these actions first (making mistakes, show confusion, etc.), recognize and support students when they also try to share their incomplete ideas, and explicitly discuss why these actions are important in helping everyone learn.

In Murata (2013) and Murata et al. (under review), the model was further developed and focused on responsive interactions within urban classrooms in the United States. Teachers can orchestrate and coordinate unique individual student learning paths by connecting and relating different ideas so that students push and stretch each other's ZPD. In planning a lesson, teachers can carefully consider the differences and the coordination of the differences, so students will learn from one another. We consider this interaction among different ideas to be a critical aspect of responsive teaching, when we maintain high academic expectations by taking advantage of diversity among students. In classrooms, different teachers' facilitation styles and different combinations of student ideas can drive classroom learning on somewhat different learning paths. Student diversity will always be there. By making these otherwise invisible student learning paths visible in planning and in orchestrating classroom discussion, we take advantage of the wealth of ideas students bring into classrooms. When we shift our focus from telling math content to students to coordinating and extending different student ideas, teaching will hold new meanings for teachers and for students, as everyone helps along the Class Learning Path to important grade-level understanding.

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