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CHAPTER 9

THE COMMON CORE MATHEMATICS STANDARDS AS SUPPORTS FOR LEARNING AND TEACHING EARLY AND ELEMENTARY MATHEMATICS

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The Clements and Sarama chapter is an excellent research summary about effective mathematics learning and teaching in preschool and the early elementary grades. This brief commentary will build on that summary to identify some ways in which the new Common Core Mathematics Standards can improve teaching and learning. Because standards do not describe learning activities, they do not describe a learning trajectory in the Clements and Sarama sense, because their term includes learning activities. Therefore, this commentary uses the term learning path to mean the experiential progression of knowledge built by a coherent appropriate program of learning activities.

Instructional Strategies for Improving Students' Learning, pages 177–186

As was emphasized in the research summary, and in all recent national reports, understanding and fluency are both crucial foci of teaching. Aspects of both understanding and fluency are mentioned specifically in the standards. Most importantly, the standards are focused and coherent across grades, and so there is time for teachers to concentrate on both understanding and fluency. At each grade level, the standards focus on fewer standards than in most previous state standards, and these standards build on each other across grades. Therefore, teachers can have enough time for grade-level mastery of core topics, and teachers of the next grade can concentrate on the goals for that grade level, as is common in other countries. Reducing the present pattern of the huge waste of time now spent in earlier grade-level reviewing (over 40% by some estimates) and the confusion about exactly which grade-level teacher is responsible for what topics will be a major positive result from the standards.

Reasoning is explicitly mentioned in the standards and such reasoning is supported by visual/conceptual aspects of the standards; therefore, age-appropriate learning paths exist. The research reviewed in the Clements and Sarama chapter indicates that it is crucial to base math teaching and learning (and math programs prepared by publishers) on learning paths (trajectories). These learning paths are particularly visible in the Common Core Standards for two of the most crucial domains for early learning: operations and algebraic thinking (OA) and number base-ten (NBT).

The OA operations and algebraic thinking standards lay out an ambitious learning path, with word problem types as the basis for understanding of operations ($+ - \times \div$). These main types of word problems (see Tables 1 and 2 on pages 88 and 89 of the Standards document) are situations in the real world that give rise to addition, subtraction, multiplication, and division. There is a huge amount of worldwide research literature on learning paths within these word problem types and on methods that students use to represent and solve such problems. The standards reflect this research literature for grades K, 1, and 2. They identify grade-appropriate levels at which students work with the various problem types and with unknowns for all three of the quantities. The standards appropriately specify that students use drawn models and equations with a symbol for the unknown number to represent the problem (*situation equations* such as $5 + \square = 8$). Thus, students will have the crucial experience with algebraic problems from grade 1 on. Algebraic problems are those where the situation equation, such as $\square + 4 = 9$, is not the same as the *solution equation*, $4 + \square = 9$ or $9 - 4 = \square$. Importantly, students also work in kindergarten with forms of equations with one number on the left (e.g., $5 = 2 + 3$ and $5 = 4 + 1$) as they decompose a given number (here, 5) and record each decomposition by a drawing or equation. Experience with these various forms of equations can eliminate the usual difficulty that U.S. students have with equations in algebra, where their limited experience with one form of equation leads them to expect only equations with one number on the right.

The operations and algebraic thinking (OA) standards outline a learning path of three levels of addition/subtraction solution methods that students use at grades K, 1, and 2: (1) direct model, (2) count-on, and (3) make-a-ten and other derived-fact methods. These levels provide a bridge between algebraic problem solving and NBT because these strategies are used in multidigit adding and subtracting also. This learning path comes right from research. Prerequisites for more-advanced strategies are identified as standards in kindergarten so that students in grades 1 and 2 can learn these strategies. Furthermore, the standards specify that subtraction is to be understood as an unknown-addend problem, and division as an unknown-factor problem. These emphasize the inverse relationships between addition and subtraction and between multiplication and division. This perspective enables programs and teachers to emphasize solving subtraction by forward methods such as counting-on to find the unknown addend; for example, $14 - 8 = \square$ is thought of as $8 + \square = 14$ and can be found by keeping track of how many counted-on from 8 to reach 14: (take away) 8, then 9, 10, 11, 12, 13, 14, so six more; or make a 10: find $8 + \square = 14$ as $8 + 2 + 4 = 10 + 4$ and $2 + 4 = 6$, so the unknown addend is 6. Forward counting methods are much easier and less error prone for children than are methods involving counting down.

The number and operations in NBT standards outline a learning path for multidigit computation based on research. Core components of this learning path are that students are

- to use concrete models or drawings and strategies based on place value and properties of operations;
- to relate the strategy to a written method and explain the reasoning used (explanations may be supported by drawings or objects); and
- to develop, discuss, and use efficient, accurate, and generalizable methods including the standard algorithm.

Thus, students simultaneously build and use understanding of place value concepts of ones, tens, and hundreds in adding and subtracting numbers that are composed of these units.

This learning path adjusts the impression given by the research summarized in Clements and Sarama about student invention of strategies versus teaching the standard algorithm first. Many of these studies present Clements and Sarama's "false dichotomy" in two ways. First, in the invent situations, students were to invent, but also they were to make sense of and discuss and explain their methods. In the teach algorithms first conditions, sense making was not necessarily a priority or even supported. Also, there are a limited number of methods for solving any problem and so most students in a given classroom do not actually invent a new method. They see it used and explained by a classmate. In a classroom where invention is stressed, students rather than the teacher model and explain methods. So what is crucial is making sense of methods and providing supports for such sense making, such as manipulatives or drawings that show tens and ones, and requiring and supporting discussion of methods. The standards require such sense making.

Second, many studies or programs calling for students to invent methods have an extended period of invention without much explicit teacher or fellow-student intervention (teaching). Teaching here is viewed as necessarily interfering with sense making by students (for more about this issue, see Fuson, 2009). Such a view can result in extended periods in which some or even many students use only primitive methods, even counting all of the objects for a 2-digit problem as late as third grade. This is detrimental to less-advanced students and is unnecessary. There is a middle view called "learning-path teaching," which emphasizes sense-making that is not traditional rote teaching and is not extended invention without help to move students to more-advanced methods. Learning-path teaching stems from major NRC reports (Donovan & Bransford, 2005; Kilpatrick, Swafford, & Findell, 2001), from the NCTM process standards (NCTM, 2000), and from research on teaching in Japan and in this country (Fuson & Murata, 2007; Murata & Fuson, 2006).

A summary of such teaching referring to aspects of these reports is given in Table 9.1. The top of the table summarizes the classroom environment supported by research that creates understanding. Details of ways to build such a sense-making math talk environment are summarized in Fuson, Atler, Roedel, & Zaccariello (2009). The Common Core Standards specify mathematical practices that are similar to the NCTM process standards and that are consistent with the learning-path teaching summarized in Table 9.1. The learning trajectory approach described in the Clements and Sarama chapter and their instructional strategies based on their learning trajectories are also consistent with this learning-path teaching.

TABLE 9.1 NRC Principles and NCTM Standards Summarizing the Class Learning Path Model

OVERALL: Create the year-long nurturing meaning-making math-talk community.
The Teacher orchestrates collaborative instructional conversations focused on the mathematical thinking of classroom members (*How Students Learn Principle 1* and *NCTM Process Standards: Problem Solving, Reasoning & Proof, Communication*). Students and the Teacher use seven responsive means of assistance that facilitate learning and teaching by all (several may be used together): engaging and involving, managing, and coaching, which involves the five subcategories of modeling, cognitive restructuring and clarifying, instructing/explaining, questioning, and feedback.

TABLE 9.1 (continued) NRC Principles and NCTM Standards Summarizing the Class Learning Path Model

FOR EACH MATH TOPIC: Use inquiry learning-path teaching and learning.

The Teacher supports the meaning-making of all classroom members by using and assisting students to use and relate (interform) coherent mathematical situations, pedagogical forms, and cultural mathematical forms (*NCTM Process Standards: Connections & Representation*) and uses *four class learning zone teaching phases* within a coherent in-depth sequence of problems and activities to help students move through their own learning paths within the class learning zone:

Phase 1. Guided Introducing: Supported by the coherent forms, the Teacher elicits and the class works with understandings that students bring to a topic (*How Students Learn Principle 1*).

a. Teacher and students value and discuss student ideas and methods ["*individual internal forms* (*IIFs*) in action"].

b. Teacher identifies different levels of solution methods used by students and typical errors and ensures that these are seen and discussed by the class.

Phase 2. Learning Unfolding (Major Meaning-Making Phase): The Teacher helps students form emergent networks of forms-in-action (*How Students Learn Principle 2*).

a. Explanations of methods and of mathematical issues continue to use math drawings and other pedagogical supports to stimulate correct relating (interforming) of the forms.

b. Teacher focuses on or introduces mathematically desirable and accessible method(s).

c. Erroneous methods are analyzed and repaired with explanations.

d. Advantages and disadvantages of various methods including the current common method are discussed so that central mathematical aspects of the topic become explicit.

Phase 3. Kneading Knowledge: The Teacher helps students gain fluency with desired method(s); students may choose a method; fluency includes being able to explain the method; some reflection and explaining still continue (kneading the individual internal forms); students stop making math drawings when they do not need them (*Adding It Up: Fluency & Understanding*).

Phase 4. Maintaining Fluency and Relating to Later Topics: The Teacher assists remembering by giving occasional problems and initiates and orchestrates instructional discussions to assist re-forming IIFs to support (form-under) and stimulate new IIF "nets for action" for related topics.

RESULT: Together, these achieve the overall high-level goal for all: Build resourceful self-regulating problem solvers (*How Students Learn Principle 3*) by continually intertwining the 5 strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition (*Adding It Up*).

Note: This is a later version of the table in Fuson and Murata (2007). This table appears in the work of Fuson, Murata, and Abrahamson (2011), in which we sought to bring together perspectives on understanding and fluency and provide language to do so. We characterized Piaget's and Vygotsky's conceptual activity as involving three types of external math forms: *situational, pedagogical,* and *cultural math forms*. We specified that each learner continually forms and reforms *individual internal forms* (*IIFs*) that are interpretations of the external forms. This parallel use of the word *forms* links the external and internal forms but emphasizes that each individual internal form may vary from the external form because the internal form is an interpretation. Doing math is using "IIFs in action" to form actions with *external forms*.

The middle part specifying the four phases of inquiry learning-path teaching is the balanced antidote to the nonproductive extremes of "invent for a long time" and traditional rote teaching. For any topic, one begins by eliciting student thinking. This might be brief for a short topic. For a major topic like single-digit or multidigit addition or subtraction, students would initially develop and use or choose their own methods in a classroom environment where they already have prerequisite knowledge and have visual supports for discussing their thinking. But soon (within a couple of days so as not to let less advanced students flounder for days) research-based mathematically desirable and accessible methods are introduced to the class if such methods have not arisen from students. These methods have been found by classroom research to be easily understood by students (more easily understood than the current common ways of writing a standard algorithm), to generalize to larger numbers readily, and to make salient important mathematical issues (e.g., moving from left to right vs. right to left or using expanded notation) that are fruitful for classroom discussion; for more details, see Fuson & Murata (2007), Kilpatrick et al. (2001), and NCTM (2011).

The Common Core Standards recognize that the methods invented by students often are generalized from counting methods and work for numbers within 100. Some methods, however, become more difficult for totals between 101 and 1000 (see details in NCTM, 2011). Therefore, the standards require students in grade 2 to add and subtract totals between 101 and 1000 (called "within 1000" in the Common Core Standards) in order to experience and generalize methods to larger numbers. Many other countries add and subtract such numbers at grade two for similar reasons.

Because the term "standard algorithm" has been such a flashpoint for the "math wars," I also wish to reiterate the crucial point made in the NRC report, "Adding It Up" (Kilpatrick et al., 2001) and in the Clements and Sarama research summary indicating that there is no single recognized "standard algorithm" for any operation. Many different forms have been used in this country and are currently being used around the world. The term "the standard algorithm" actually refers to the major mathematical features of the process and not to the details of how these are written. For example, multidigit addition and subtraction have two components: (a) adding or subtracting like units; and (b) when needed, group ten of a unit to make one of the next-left unit or ungroup one unit to make ten of the next-right unit. There are more and less accessible ways to write these steps, and students should see and be able to use more accessible versions. These are often minor variations that produce major decreases in errors and increases in understanding. Figure 7.3 in the Clements and Sarama chapter shows a subtraction standard algorithm that is much easier for students because they concentrate on any needed ungrouping first and then do all of the subtracting. Figure 9.1 here shows an easier addition standard algo-

	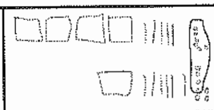	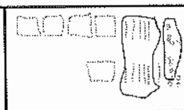	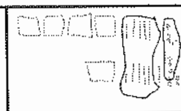
456 +167	456 +167 1 3	456 +167 1 1 23	456 +167 1 1 623
	Add 6 ones and 7 ones, write thirteen (one, ten, three ones), with the three in the ones place and the 1 ten *under* the tens column.	Add 5 tens and 6 tens to make 11 tens, and 1 more ten makes 12 tens. Write 12 tens, with the 2 tens in the tens column and the 10 tens (100) *under* the hundreds column.	Add 4 hundreds and 1 hundred to make 5 hundreds; add 1 more hundred to make 6 hundreds.

Figure 9.1 A mathematically desirable and accessible addition algorithm.

rithm. Writing the new tens or hundreds below in the next-left place makes it easier to add the numbers in those places (you can just add the two numbers you see and increase that total by one), shows the 2-digit totals clearly because their numbers are close to each other (13 ones and 12 tens), and allows students to write a teen number in their usual way (13 as write 1 then 3, not write the 3 and carry the 1). Both figures show how drawings of hundreds, tens, and ones can support the BAMT (break apart to make ten) methods by grouping numbers within 5-groups that show how much more to make a ten.

Because so much of the Common Core Standards reflect learning progressions and specify general kinds of visual supports for meaning making, these progressions and visual supports will need to be in standards-based math programs. Therefore, professional development (PD) can be more successful because more programs can be used as the basis of PD because more programs will contain many of the features described for successful PD at the end of the Clements and Sarama chapter.

The final two points concern the nature of the crucial bases for sense making in the classroom, visual teaching/learning supports. One important function of national reports and of the Common Core Standards has been to stimulate the use of such visual teaching/learning supports connected to mathematical notation and reasoning because so much research supports such use. However, sometimes crucial supports do not even exist. In such cases, reports and the standards can identify the need for them. Most of the illustrations given in the geometry learning trajectory in the Clements and Sarama chapter are not about the central 2-D shapes used in geometry (e.g., right-angled shapes such as squares, rectangles, and right triangles, which form the basis for conceptualizing area as the number of contiguous squares and for finding the formulas for the area of most shapes used widely in geometry). Only two illustrations used at ages 7 and 8 use squares, rectangles, and right triangles. Most earlier illustrations used shapes made from equilateral triangles. This is because physical sets of such shapes (often called "pattern blocks") have been available and widely used for a long time. It was the National Research Council's report on early childhood math, *Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity* (2009), that identified the need for such materials for preschool and early elementary students. Students can engage in all of the levels in the geometry progression with a right-angled set of pattern blocks made from a few key squares, rectangles, and right triangles all based on a square of one inch. This enables them to build important informal knowledge leading to many of the Common Core geometry and measure standards.

Finally, teaching/learning supports and solution strategies interact with the language the child is speaking. The BAMT method shown in Clements and Sarama's Figure 7.1 is an excellent general method that can be viewed (and is viewed in East Asian countries) as the first step in multidigit adding and subtracting (you are already grouping or ungrouping). However, this method is more difficult in English than in East Asian languages, where teen words are said in a regular form such as *ten two* for 12. When 8 + 6 is recomposed to be 10 + 4, that is said as *ten four* in regular East Asian words, and so the conceptual step from *ten and four* to *ten four* is small (but not nonexistent; see Ho and Fuson, 1998). In English, children must take a larger step from *ten and four* to *fourteen* or, even more difficult, *ten and two* to *twelve*. So they must become fluent in all of the different numerical examples for teen numbers; the importance of this conceptual work is recognized in the kindergarten standards.

Likewise, the empty number line shown in Clement and Sarama's Figure 7.2 is used in The Netherlands with a language that says all 2-digit words with a reversal of the tens and ones. The example of 85 – 68 shown there would be said by the children as *five and eighty* minus *eight and sixty*. Therefore, a counting method and visual counting support that keep together the two-digit number is quite useful in dealing with the reversals between the written and spoken 2-digit numbers. Such methods are less necessary in English, where children can use counting-on methods by using math drawings of tens and ones to keep track of how many are counted on (NCTM, 2011; Fuson, Smith, & Lo Cicero, 1997). Of course, forward methods can also be supported by any visual supports: Many children find it easier to find 85 – 68 = ☐ by adding on ones and tens to 68 to make 85 (68 + ☐ = 85) using the concept of subtraction as finding the unknown addend, as described in the Common Core Standards.

Because the Clements and Sarama chapter focuses on preschool and early elementary mathematics, this commentary closes with a reminder about the crucial importance of the research summaries and recommendations of the NRC report on early childhood math of 2009 for creating a national environment of learning in the preschool years so that all kindergarten children can meet the Common Core Standards. Teaching approaches that implement these NRC recommendations are given in the books for teachers published by the National Council of Teachers of Mathematics (2010a, 2010b, 2009, 2011). Use of such research-based approaches in educational and care centers can help close the equity gap and can help all children achieve at a high level in early and elementary mathematics.

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