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# Complexities in Learning Two-digit Subtraction: A Case Study of Tutored Learning

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This case study of a low-achieving first grader learning to subtract two-digit quantities with several different pedagogical objects demonstrates the complexities of the conceptual shift from the tens part of a number to the ones part. Perseverations (failures to shift) occurred with the quantities, the count words, and the number in the tens or ones position. These difficulties reappeared in each new context and in a previously successful context. Over sessions, overcoming them gradually required less help from the first-grade tutor and the supporting adult tutor. Conceptual structures required for and a component analysis of two-digit subtraction are described. The range and nature of the learning difficulties described indicate the complexity of the learning task facing children, especially those speaking European languages. The success of the first-grade tutors indicates that classmates can be taught to provide excellent learning support.

Multi-digit subtraction is considerably more difficult than multi-digit addition, and many second- and third-graders in the United States and Europe do not carry it out correctly (Bednarz & Janvier, 1992; Beishuizen, 1993; Fuson, 1990, 1992a, 1992b). Subtraction is particularly prone to a very frequent error in which the smaller number is subtracted from the larger, regardless of whether

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the larger number is on top (VanLehn, 1986; Fuson, 1990, 1992a). This error results from thinking of multi-digit numbers as concatenated single-digit numbers instead of as multi-unit quantities (Fuson, 1990).

Conceptual structures for two-digit numbers that allow children to carry out and understand multi-digit subtraction have been identified in a series of papers (Fuson, 1990; Fuson et al., in press; Fuson, Smith, & Lo Cicero, submitted). These conceptual structures are much more complex for children speaking European languages than for children speaking Asian languages with number words that are based on Chinese. European languages have various irregularities for two-digit numbers, whereas Chinese-based languages are regular and explicitly state the quantities of ten in those numbers (52 is said as "five ten two" and 13 is said "ten three"). Therefore, the teaching and learning tasks for European teachers, parents, and children are more challenging, complex, error-prone, and potentially prolonged than are those tasks for Asian teachers, parents, and children.

In this paper we elucidate these learning tasks for two-digit subtraction by examining examples from a case study of a first grader involved in a mathematics intervention study in an urban school in the United States. This child had only partially constructed the requisite web of related place-value conceptual structures and was one of five children at the bottom of the mathematics class. We used the two-digit subtraction context as a setting for constructing multi-digit conceptual structures as well as methods for carrying out subtraction. Breakdowns during two-digit subtraction may thus result from a lack of subtraction understanding or from missing or incorrect aspects of the multi-digit conceptual structures. Many children constructed much of the requisite subtraction and place-value knowledge through classroom activities, and some children did so exceedingly rapidly. However, the least advanced children were not able to make these constructions within the busy and complex environment of an urban school classroom. These children were tutored over as many as 11 sessions extending over the last 14 days of school in June. In-depth case studies were done of three children tutored over at least 9 sessions each (Beschorner, Sartini & Taniguchi, 1995). Watching the constructions and breakdowns over time in these children is like watching a slow-motion film of faster learners and helps us to understand the complexity of the attentional and conceptual demands of this situation.

The tutoring was done in a peer-tutoring format, with an adult teaching about tutoring and then supporting the tutoring of the target child by the peer. The four top first graders in the class had very good understanding of multi-digit numbers and of two-digit subtraction. We wanted to explore the extent to which they could learn to help these children at the bottom of the class with the whole range of errors they make. We had found in earlier work (Fuson et al., submitted) that adult tutoring of these children needed to be individual rather than in small groups, because each child had different missing knowledge, and the attentional demands of the situation were so high that distractions were extremely disruptive. The dyadic structure of individual tutoring, with immediate constant access to support and no distractions by other children, was an effective learning situation for the very lowest-achieving children in a class. We wanted to explore whether children in the class could serve as tutors, because children are readily available whereas adults may not be. This peer-tutoring model did require initial adult teaching and support of the peer tutors, who were otherwise likely to do the problem for the child or just tell them the answer (Smith & Fuson, in preparation b).



This particular case study was chosen to show in depth a child's learning of one of the most difficult but essential aspects of multi-digit thinking: shifting from quantities of ten to quantities of one. This shift is required for any adequate multi-digit conceptual structure and for any conceptual method of subtraction other than counting by ones, and it must be done at least three times during a subtraction problem. Some other aspects of learning necessary in this domain are also touched on in this case study, with an overall goal being to increase the reader's appreciation of the special difficulties encountered by children using European number words. A secondary purpose is to give some sense of the peer-tutoring capacities of expert first graders after experiencing tutoring support.

The paper is separated into the following sections: First there is an overview of conceptual structures in this domain. Then the pedagogical supports used to help children construct adequate conceptions in our intervention classroom and tutoring sessions are described. Our analysis of elements of two-digit subtraction carried out with these pedagogical supports is presented, and then the theoretical perspective, empirical background, and methods are described. The case study itself is presented by intertwining transcriptions of key interactions with discussion of these transcriptions. Results and implications of the case study are then summarised.

## CONCEPTUAL STRUCTURES FOR TWO-DIGIT NUMBERS

A developmental sequence of children's conceptual structures for two-digit numbers is illustrated in Fig. 1. Each conceptual structure consists of three elements (a quantity; its spoken name as a number word; its written symbol as a number mark[1]) and the six possible connections between these three elements. This structure is quite simple in the unitary single-digit conception at the bottom right of the figure. For the numbers 1 to 9, a child must be able to make all six of the arrow connections shown in the triad: from the word "five" make a

[1]We use the word "mark" to remind the reader that the meaning of a mark depends upon the available meanings for each viewer of the mark; children's meanings may differ from the meanings of a reader.

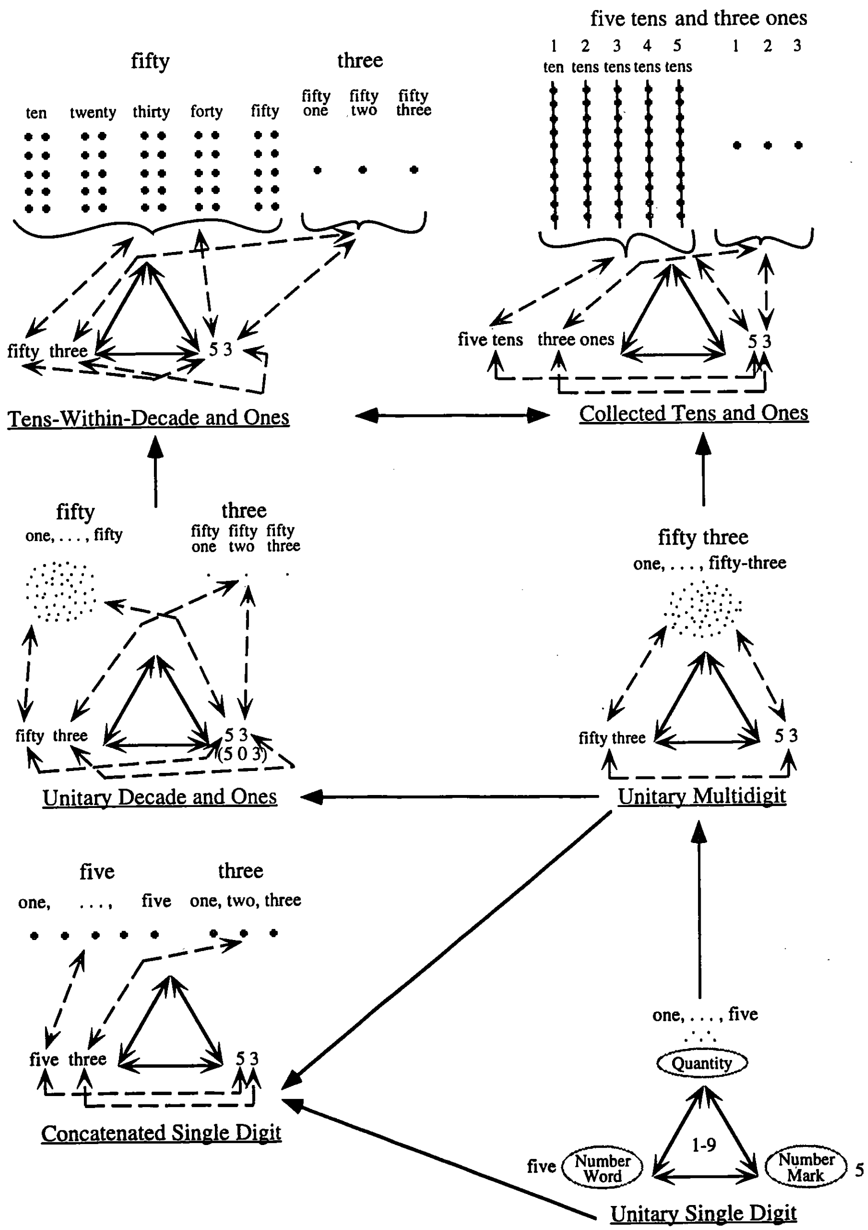


FIG. 1. A developmental sequence of children's two-digit conceptual structures (from Fuson et al., in press).





quantity of five and make the number mark 5, from 5 objects label them with the word "five" and with the number mark 5, and from the number mark 5 say the word "five" and make five objects. The associations between the number words and number marks are largely rote associations (though some systems for making these relations quantitative have been used), and the links between a number mark and a quantity are ordinarily carried out by using the number word and counting the quantity (i.e. using the bottom horizontal and the left path rather than the direct path), at least for numbers of 5 or more (unless a pattern arrangement is used).

Children then extend this unitary single-digit conception to a unitary multi-digit conception in which the same six connections are made and there is no special meaning attached to each numeral in the two-digit number mark. So 13 just means a pile of thirteen objects, and the 1 and the 3 have no special significance for the quantity, and "thirteen" is just a number in the counting sequence that comes after "twelve" and before "fourteen". Many children are also drawn by the visual appearance of two-digit written numbers and form a concatenated single-digit conception of numbers in which each digit takes its meaning from the unitary single-digit conception, and these single-digits are just considered to be written beside each other (bottom left of Fig. 1). So, for example, in C. and M. Kamii's task (C. Kamii, 1985, 1989; M. Kamii, 1982) a child seeing 16 will count out sixteen objects but will show six of them as the meaning of the 6 and one object as the meaning of the 1. There is no sense that the 1 means one group of ten objects. The use of this concatenated single-digit conception is quite pervasive in children speaking European languages (e.g. see reviews in Fuson, 1990, 1992a, 1992b; Sinclair, Garin, & Tieche-Christinat, 1992). Even when children have constructed one of the more advanced conceptions, the constantly seductive nature of the appearance of written number marks as single digits elicits this concatenated single-digit conception in children, suggesting common multi-digit subtraction errors.

The two conceptions at the middle left and upper left are the result of the special decade words in European languages. Most such languages do not explicitly say "two tens" or "six tens" as they do for the next quantities, hundreds and thousands (e.g. two thousand six hundred). Instead, there is a special list of decade words that originally may have been more explicit concerning the number of tens but now show long-term effects of pronunciation changes to facilitate easy pronunciation rather than conceptual clarity (Menninger, 1958/1969). Details for several European languages are discussed in Fuson and Kwon (1991/1992). This special list of decade words (ten, twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety in English) must be learned. In the United States, this may take eighteen months or even longer (e.g. Fuson, Richards, & Briars, 1982), and some third graders still cannot count to one hundred by ones or by tens (Maine, 1995). Children fairly readily learn the pattern of decade chunks: "*x*ty, *x*ty-one, *x*ty-two, . . . , *x*ty-nine, *y*ty, *y*ty-one, . . . , yty-nine, zty", etc. But they have a great deal of difficulty, and relatively little directed support, in learning the list itself. In contrast, children speaking Chinese have a much easier task, and this is reflected in faster learning than in either English or Italian (Miller, Agnoli, & Zhu, 1989; Miller & Stigler, 1987; Miller & Zhu, 1991).

After learning the special decade list, children speaking European languages have two special conceptual structures to construct: the *unitary decade and ones* and the *tens-within-decade and ones* conceptions (middle and upper left in Fig. 1). In the unitary decade and ones conception, children construct the cardinal meaning of fifty-three as fifty and three, as a quantity of fifty plus a quantity of three. They also attempt to connect each number word with a part of the number mark: fifty with the 5 and three with the 3. At this point, this conception may lead them to a typical error, writing fifty-three as 503 (50 and then 3), using concatenation in space to show the concatenation in time of the number words. Without some sense of the fifty (= 50) hiding behind the 53 ("seeing" an invisible 0 under the 3 so that 5 is seen as 50 plus 3 more), children with this conception have difficulty learning the written mark 53 as fifty-three because they actually write a 5, not a 50. We did use to considerable effect the notion of an invisible 0 hiding behind the 3 (we wrote the 0 in faint dots and then wrote the 3 over it) in classroom interventions with first graders (Fuson et al., submitted). Montessori cards in which the 3-card is placed on top of the 0 in the 50-card to show 53 also use this unitary decade and ones conception.

Children who can count by tens and by ones (e.g. 10, 20, 30, 40, 50, 51, 52, 53) and who have opportunities to count objects grouped in ten in this way can construct a tens-within-decade and ones conception in which a two-digit quantity is considered as built up by groups of ten and by ones. To understand the written digits, this conception requires a field-ground shift from viewing the groups of ten objects as fifty ones (the sequence word "fifty" meaning that many ones) to seeing them as five groups of ten (the 5 in 53 can now have its single-digit surface meaning of "five" but as five groups of ten). While thinking "fifty three", a child may still be subject to the 503 error, but the understanding and seeing of the 5 groups of ten can give an alternative meaning to the number mark 53 as 5 groups of ten.

Children speaking Asian languages with regular-named tens do not have to construct either of the two decade and ones conceptions. They move from the unitary multidigit conception to the collected tens and ones conception, which is supported by their number words. If these children have experience with objects grouped into tens, there is a simple match between the object groupings, the number words, and the written number marks (see the top right-hand triad in Fig. 1). They count and see five groups of ten, say "five ten" (Chinese does not have plurals) and write 5 in the tens place. They then count or see three single objects, say "three", and write 3 to the right of 5. They must shift from the groups of tens to the single objects, and consider the tens first in all three domains, but the connections are all fairly straightforward. If objects are not grouped in tens, counting objects suggests such grouping because the number words separate the counted objects into groups of ten and single ones (. . . , ten, ten one, . . . , ten nine, two ten, two ten one, . . . , two ten nine, three ten, three ten one, . . .). Thus, the number words themselves may suggest to children that they should arrange objects into groups of ten as they are counted, and the number words make it easy to notice and understand any groups of ten that are already there.



In the United States, in most traditional mathematics classrooms the only part of the collected tens and ones conception that is supported is the connection at the bottom of the triad between the number words and the written numbers. Children do pages of work-sheets filling in the blanks to learn the patterns "___ tens and ___ ones is 53" and "5 tens and 3 ones is ___". These exercises are rarely linked to quantities grouped in tens, so many children have difficulty constructing any conception of two-digit quantities as consisting of multi-units of tens and ones. Thus, for such children, no such conception is available to support comprehension of two-digit subtraction as operating on quantities. This situation stems from the dominance of textbook use in traditional US classrooms. Teachers either let children move at their own pace through pages, or the whole class works as individuals on specified pages each day. The economic importance of the adoption of particular textbooks by large urban districts leads textbook companies to conform to most state and urban guidelines, with a remarkable uniformity of topic coverage that amounts to a national curriculum (see Fuson, 1992a, and other chapters in Leinhardt, Putnam, & Hattrup, 1992).

Children who have an opportunity to construct both a tens-within-decade conception and a collected-tens conception may go on to integrate these two conceptions within an integrated multi-unit conception shown as a triangular prism in Fig. 2. Each of the elements of the triad within each conception becomes bidirectionally linked so that a child may rapidly and flexibly move from any point in one conception to any point in the other. The fifty and the five tens become flexibly linked as quantities, enabling rapid shifts from fifty as five groups of ten ones to five single tens. These shifts occur for quantities and for words (i.e. "fifty is five tens"). The child tutors in the present study used such integrated multi-unit conceptions in their tutoring.

In Fig. 3 the three conceptions that go beyond a unitary multi-digit conception are shown in the connected web that must be constructed by a given child who learns all three conceptions (the unitary conception, the tens-within-decade and ones conception, and the collected tens and ones conception). The collected tens and ones conception is shown in heavy lines as the centring conception. This conception has five paths (a, b, c, d, e). The i and ii written to the left of Paths b and c show the knowledge necessary for children to take those paths (count

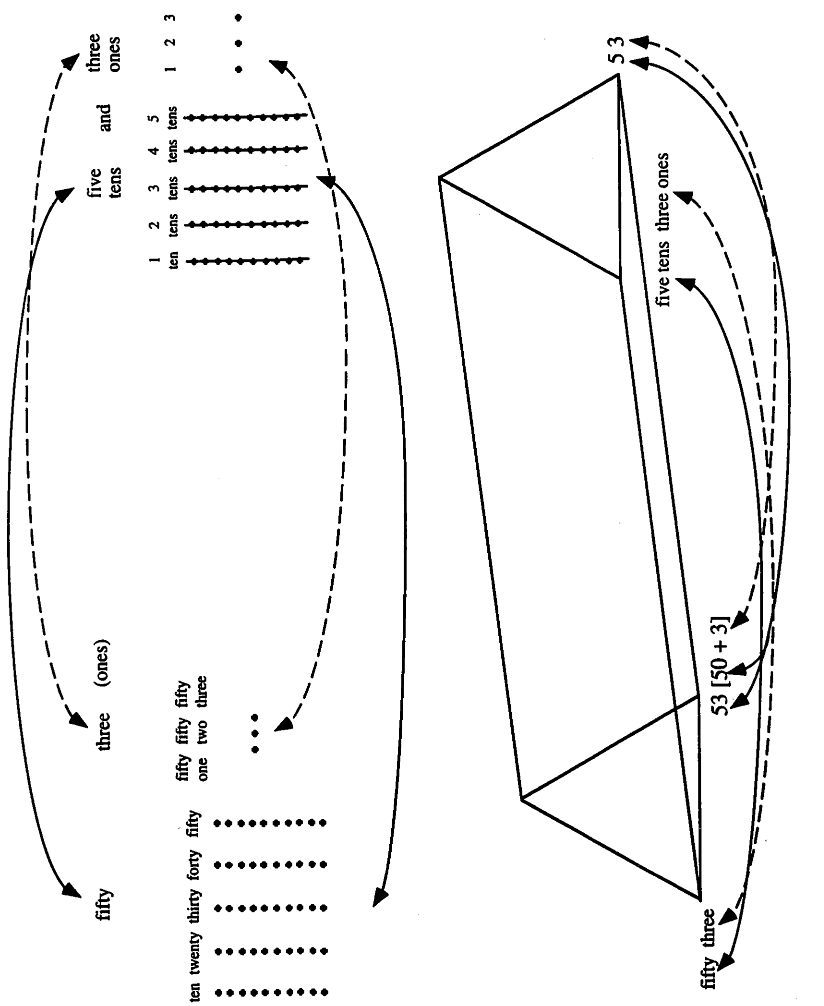


FIG 2. Integrated multi-unit conception.





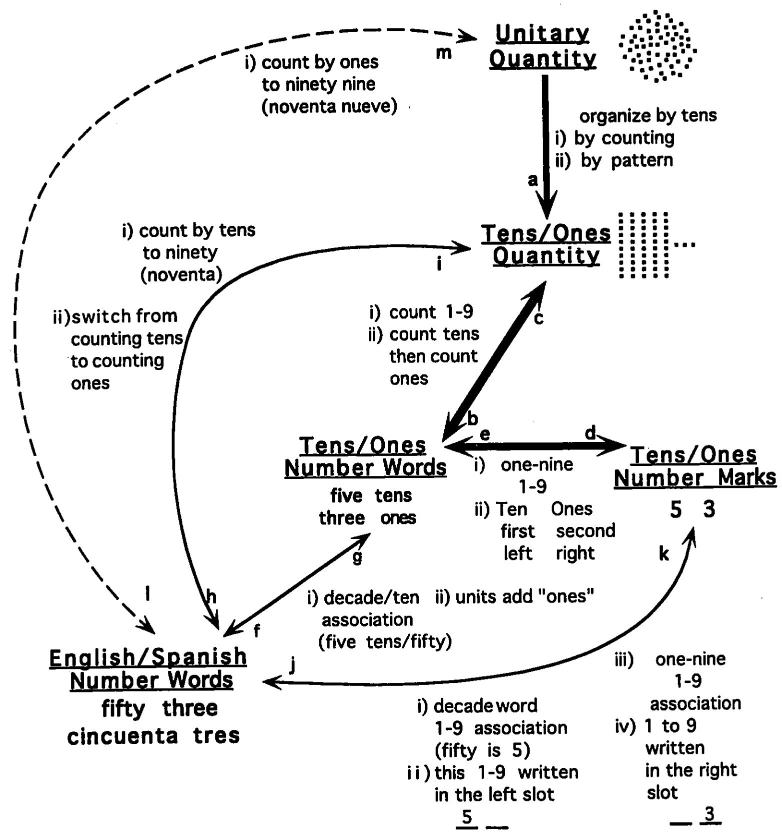


FIG. 3. Quantity, number mark, and number word relationships for two-digit numbers (adapted from Fuson et al., submitted).

objects from 1 to 9, count the tens groups first by ones and then count the single units by ones). The i and ii written below Paths d and e show the knowledge necessary for those paths (associations between each pair "one/1, two/2, . . . , nine/9", saying the tens number first and writing that digit on the left and saying the ones number second and writing that digit to the right of the tens digit). Children must also be able to group quantities into groups of tens and the left-over ones; this is shown as Path a. These heavy lines and their requisite knowledge are what must be learned by Asian children: five paths and with their moderate amount of knowledge of relationships.

All of the rest of Fig. 3 is special knowledge required by the European languages. This drawing does not even contain all of the special problems with words between 10 and 20 that many languages including English have. For example, fourteen and forty sound very much alike, but they mean "one ten four" and "four tens one" and are written 14 and 41: The teens are said in reverse order from the decade words, with the ones said first rather than second. We do not consider those problems here, but they are discussed in Fuson and Kwon (1991/1992), as are special problems with several other European languages. The upper-left dotted line (Paths l and m) in Fig. 3 is the unitary conception in which a quantity is counted by ones. The solid Paths i and h are the decade counting by ten part of the tens-within-decade conception, and the path f/g is the background/foreground shifting between the number of groups of ten (5) and the number of entities in those groups (fifty). The bottom-right solid line (Paths j and k) involves the decade word/written number links; their required knowledge is given below the paths in i, ii, iii, iv. Figure 3 shows a formidable web of knowledge that must be constructed by children if they are to be able to solve problems using a tens-within-decade and ones conception (e.g. count by tens and count by ones, eventually leading to counting on by tens and ones) and a collected tens and ones conception. Alternatively, one could construct a classroom that centred on the heavy lines and fostered, at least at first, the simpler regular collected tens and ones conception.

## PEDAGOGICAL SUPPORTS FOR CONSTRUCTING TWO-DIGIT CONCEPTUAL STRUCTURES

The class of first graders involved in the case study had built up their knowledge of two-digit numbers by using the approach described in Fuson et al. (submitted) of drawing quantities as vertical columns of ten dots with any extra dots made horizontally to the right. Children initially worked with objects, collecting ones into tens. Then they began drawing ones collected into tens as columns of dots. For example, 53 was shown as 5 columns of ten dots and 3 horizontal dots to the right. Eventually a line segment was drawn through a column to connect the ten dots to show each ten more clearly, and finally only these ten-sticks were drawn, without showing the ten dots within them. Examples of subtraction using these drawn quantities are given in Fig. 4. The initial quantity is drawn, the tens and ones to be subtracted are circled or crossed out, and the remaining quantity is counted. In problems, such as those shown, in which more ones must be taken away than are explicit in the initial quantity, the ones inside a ten-stick must be "seen" (i.e. a ten-stick must be conceptually changed from being one ten to being ten ones) so that they can be taken away. The method shown in Fig. 4 is a common method used by many children in this class. They circle the part of the ten they are taking away, and make dots to show the ones that remain. The top right-hand problem was done by the peer tutor, who explicitly connected her taken-away (circled) quantities to the number. The bottom two examples were done late in tutoring by the tutee.



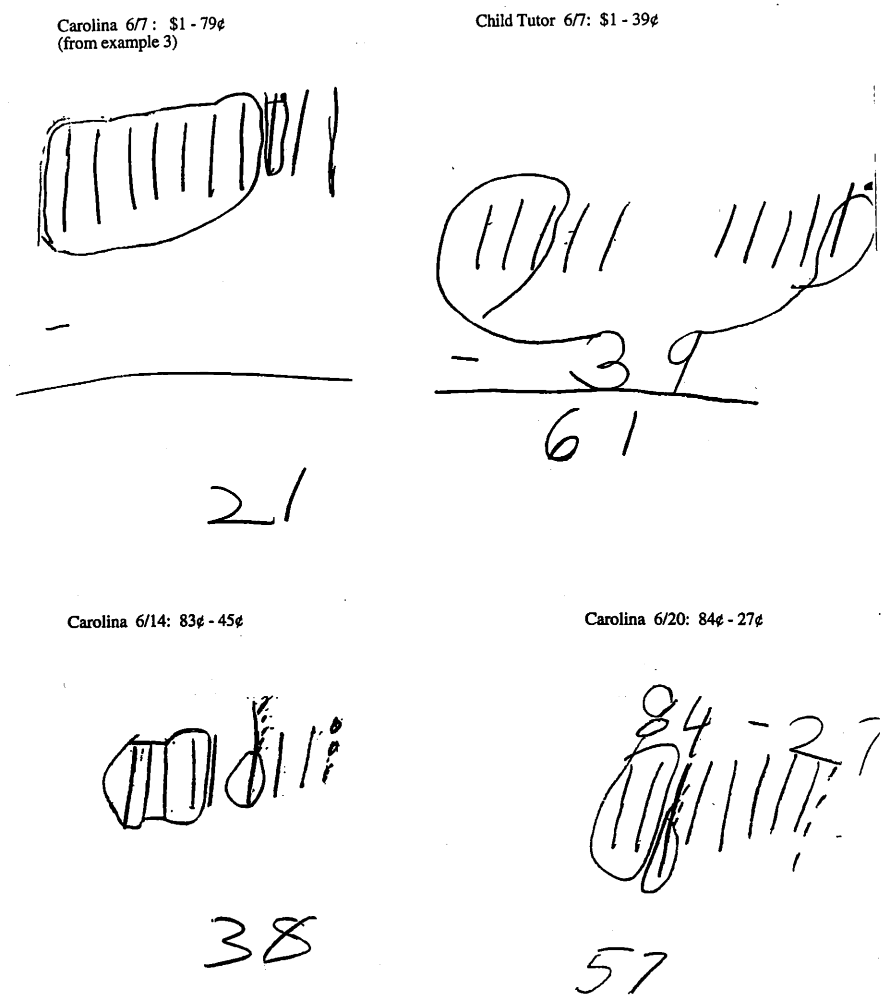


FIG 4. Subtraction using ten-sticks and dots.

In early June children in this class began to use penny-strips[2] made of lightweight cardboard (see Fig. 5). Ten pennies were shown on each strip, with a space between two groups of 5 pennies so that the two fives could easily be seen. The pennies were oriented on the strip so that the strip would be a vertical column of pennies, like the dot drawings. On the back of each strip was a dime,

[2] A penny is the 1-cent (¢) coin in the United States. A dime (smaller than a penny and shiny silver rather than copper or bronze, as pennies are) is the 10¢ coin, and a nickel (larger than a penny) is the 5¢ coin. There also is a 25¢ coin (a quarter), which we did not use.

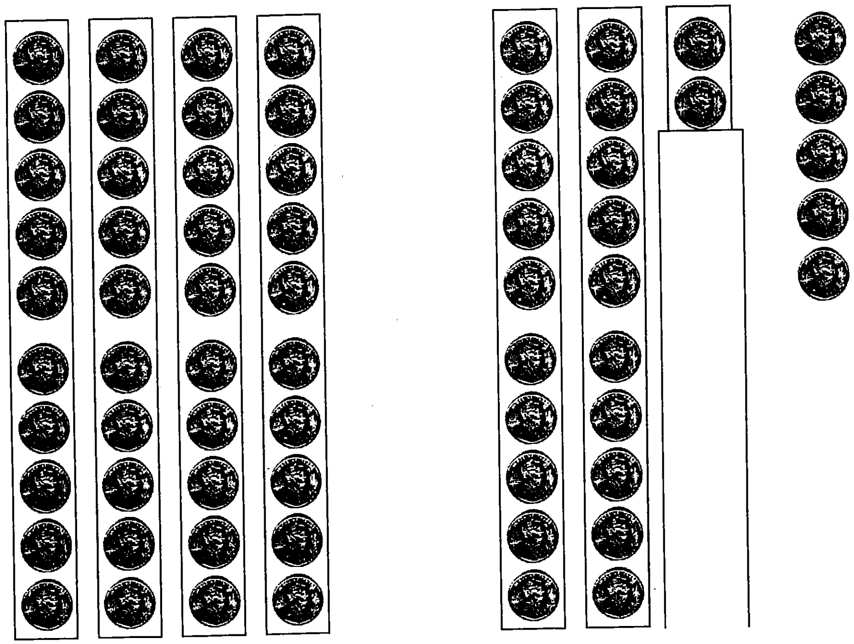

FIG. 5. Subtraction using penny-strips. Carolina's solution in Example 4: She has 75¢ which she makes with 7 penny-strips and 5 pennies, pays 48¢ by separating (taking away) 4 tens, then covers (takes away) 8 ones from a ten with a piece of paper, then counts the rest (2 penny-strips and 7 pennies = 2¢).

showing the quantitative relationship between pennies and dimes. Penny/nickel strips showing 5 pennies on one side and a nickel on the other were also made. Two nickel-strips fit on top of one dime-strip, showing those relationships in pennies, nickels, and dimes. Real pennies were used as the units with the penny-strips. The penny-strips were invented because work with first and second graders during the year had indicated that many children had considerable difficulty with understanding money, and they needed quantitative supports for constructing the necessary relationships (Fuson, Zecker, Lo Cicero, & Ron, 1995). The penny-strips were designed so that they could be a precursor to the sticks-and-dots drawings. They had the advantage that it was as easy to make tens as to make ones: a child put out a strip as s/he counted each ten (one ten, two tens, three tens, four tens). Thus, with the penny-strips, children could begin to understand and operate with tens that showed the ten ones within the ten (the ten-stick did not). The penny-strips also could be counted in sequence words (ten, twenty, thirty, forty), so they could support the whole multi-digit web.



The penny-strips were invented to overcome money-learning difficulties identified (mostly in other classes) while and after this class had learned tens-sticks and dots (Fuson et al., 1995). The penny-strips share characteristics of base-ten blocks in that both show a ten as one object containing ten ones. Base-ten blocks were invented by Zoltan Dienes (1960, 1963) and have been used effectively in classroom and small-group studies to help children learn multi-digit addition and subtraction with understanding (e.g. Dienes, 1963; Fuson, 1986; Fuson & Briars, 1990; Burghardt & Fuson, submitted). We invented the penny-strips explicitly to teach money and because base-ten blocks are too expensive for impoverished urban schools.

Figure 5 shows the method of subtracting the penny-strips that children used in the tutoring sessions.[3] A covering strip was used to cover pennies that were taken away from a ten-strip. The initial quantity (here 75¢) is made with 7 penny-strips and 5 pennies, the purchased quantity (48¢) is taken away (physically taking away 4 ten-strips—here they have been moved to the left away from the others—and covering—taking away perceptually—8 ones). Then the remaining quantity is counted. For problems in which the number of ones to be taken does not exceed those explicitly in the initial quantity, the ones can also be taken away physically.

Eventually the penny-strips can be replaced by dimes. Figure 6 shows subtraction with quantities assembled from dimes and pennies. Here both tens and ones can be physically taken away. For problems made with all dimes and no groups of 10 pennies, and in which there are not enough ones, a dime must be traded for 10 pennies in order to get enough pennies to take away.

The function of each of these quantitative pedagogical supports is to help children construct and use, for different referents, the web of two-digit knowledge shown in Fig. 3. Each pedagogical support has different ways of showing two-digit quantities grouped into tens; children then learn to use their number words and number marks to describe various such quantities. For the dimes and pennies, the goals also include understanding and being able to use money in real-world situations. The pedagogical objects can be used to subtract two-digit quantities. The aim is to link such subtraction to subtraction written with two-digit numbers so that these numbers can take on the meanings of the quantities. Eventually subtraction will be done just with numbers, but in a meaningful way, not as subject to the intrusions of errors from the constantly seductive digits with their concatenated single-digit conception.

[3]Most children in the class folded the penny-strips to take away ones, but some became confused about which part they had taken away; therefore, the covering strip was tried in the tutoring, and it worked well.

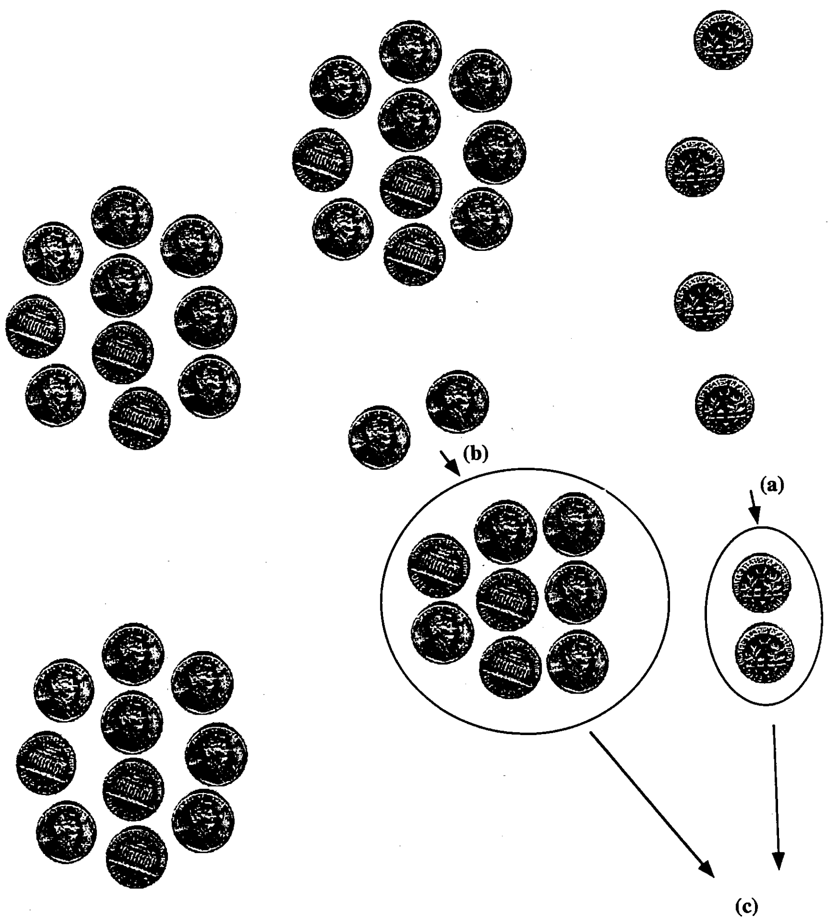


FIG. 6. Subtraction using coins: Carolina's solution in Example 12. Constructs $1 in 6 dime and 4 ten-penny groups, separates 2 dimes (a) and 8 pennies (b), pulls these coins away (c) to take them away, and counts the rest.

## KNOWLEDGE REQUIRED FOR TWO-DIGIT SUBTRACTION WITH QUANTITIES

Typical subtraction methods for each kind of pedagogical support were briefly described in the previous section and shown in Figs. 4 through 6. Different methods were used in the class, and individual children had their own variations of particular methods; but overall these methods, and the knowledge used in these methods, can be organised and summarised as shown in Fig. 7.



The knowledge on the left-hand side of Fig. 7 concerns the subtraction situation itself and the understandings of the role of each number in the subtraction real world (and written number) situation. A start number must be identified, a take-away number that is to be taken away from the start number must be identified, and then a number that describes what remains after the taking away must be found. Each of these numbers is operated on as quantities made up from the available pedagogical objects; these operations on quantities are given in the corresponding right-hand sections of Fig. 7. All of the methods used in our case-study class were take-away methods that directly model the taking away in the situation: Quantities are *taken away from* the *original* quantities to leave the *rest*. Two-digit subtraction can be modelled using objects such as penny-strips or base-ten blocks in other ways. Children who have used comparison meanings of subtraction can make quantities for both the known numbers and compare them to find out how much more one number is (e.g. Fuson, 1986), or children may use quantities to model the three numbers in a written method (e.g. Burghardt & Fuson, submitted). However, the children in our class all directly modelled the take-away buying situations used in this case study by taking-away methods, as did the children in the Resnick and Omanson (1987) tutoring study with base-ten blocks. We have found, and Resnick and Omanson described the same issue, that children lacking strong comparison experience who make quantities for the second number as part of "setting up" the subtraction problem may then make errors in subtracting: They either subtract the second quantity, leaving the start quantity as the answer, or they add the quantities because they see two of them.

The right-hand sections of knowledge in Fig. 7 concern triad relationships among quantities grouped in tens and ones, sequence and/or tens and ones number words, and written two-digit number marks. The top right-hand section, section A, describes the making of the start number in tens and ones quantities. The middle right-hand section, section B, describes the taking away of quantities for the take-away number. The bottom right-hand section, section C, outlines the counting of the remaining quantities to find the rest number (the answer to the subtraction). Carrying out the subtraction requires moving after each step from the knowledge about the problem situation to knowledge about tens and ones quantities or vice versa, as indicated by the horizontal arrows.

There are different ways in which children work within each quantity section. A major difference is in whether a child makes, subtracts, and counts using tens and ones words or using sequence (English or Spanish) number words. Both kinds of strategies are described in Fig. 7 for making, subtracting, and counting quantities. The difference between these two strategies is indicated in Fig. 8. With the tens/ones number words, a child sees groups of tens and counts the groups by ones: "One (group of ten), two (groups of ten), three (groups of ten), four (groups of ten), five (groups of ten)." The child then shifts to the ones quantities, sees each single quantity, and counts them by ones ("one, two,

IDENTIFY PROBLEM AS A TAKE-AWAY PROBLEM (e.g. 65-37)
  [if problem given orally, MD Words → MD Marks]
  write problem #s to remember them

IDENTIFY START # (65) IN PROBLEM →
  know need to construct START # with objects
  [inhibit previous START #s and groups of objects present]
    a. in penny-strips
    b. in sticks and dots
    c. in dimes (or stacks of ten pennies) and pennies
    d. in vertical marks form

A. MAKE TENS/ONES QUANTITY REPRESENTATION (of 65)
  *Make tens quantity*
    identify tens single-digit # (6)
    choose tens quantity objects (strips, sticks, stacks, dimes)
    count six tens objects (groups of ten)
      T/O Words: count by ones
        while remember/monitor/stop at six
      Sequence Words: six to sixty (SD Mark/Word → Sequence Word)
        and count by tens
        moving *one* tens group each time
          while remember/monitor/stop at sixty
  *Make ones quantity*
    identify ones single-digit # (5)
    choose ones objects (pennies, dots)
    count five ones objects
      T/O Words: count by ones
        while remember/monitor/stop at five
      Sequence Words: count on by ones from sixty
        while remember/monitor/stop at sixty five

← IDENTIFY TAKE-AWAY # (37) IN PROBLEM
  know to take away TAKE-AWAY objects from START objects
  [inhibit addition construction of a 37]
→

B. SUBTRACT TENS/ONES QUANTITY (37)
  *Take away tens quantity*
    identify tens single-digit # (3)
    choose tens quantity objects (strips, sticks, stacks, dimes)
    take away/cross out/circle/cover three tens objects (groups of ten)
      T/O Words: count by ones
        while remember/monitor/stop at three
      Sequence Words: three to thirty (SD Mark/Word → Sequence Word)
        and count by tens
        taking *one* tens group each time
        while remember/monitor/stop at thirty
  *Take away ones quantity*
    *if* TAKEAWAY ones > START ones, must first take some ones
      from a ten: one ten becomes ten ones by a conceptual shift or
      by trading for/drawing ten ones
    identify ones single-digit # (7)
    choose ones objects (pennies, dots)
    take away/cross out/circle/cover seven ones objects
  T/O Words: count by ones
    while remember/monitor/stop at seven
  Sequence Words: count on by ones from thirty
    while remember/monitor/stop at thirty-seven

IDENTIFY THE REMAINING OBJECTS AS THE REST →
  Find the REST # (= 28)

C. COUNT REMAINING TENS/ONES QUANTITY (= 28)
  *if* original ones left and ones left in the ten, counts both
  *Tens/Ones Words* (i and ii in either order or iii) *method*
    i) count tens objects by ones
       and write two [to left of 8 if i is second]
    ii) count ones objects by ones
       and write eight [to the right of 2 if ii is second]
    or iii) count tens objects by ones
       count ones objects by ones
         while remember two tens
       write two tens eight ones:
         write 2
           while remember eight
         write 8 to the right of 2
  *Sequence Words method*
    count tens objects by tens
    count on the ones objects
      by ones from tens count result (20)
    write twenty eight (Sequence Words → MD Marks):
      twenty → 2 (single digit #)
        write 2
          while remember eight
        write 8 to the right of 2

← CONSIDER ANSWER WITHIN THE PROBLEM SITUATION
  (twenty-eight, or two ten eight, what?)

FIG. 7. Multi-digit take-away problem component analysis. MD = multi-digit; SD = single-digit; T/O = tens/ones.

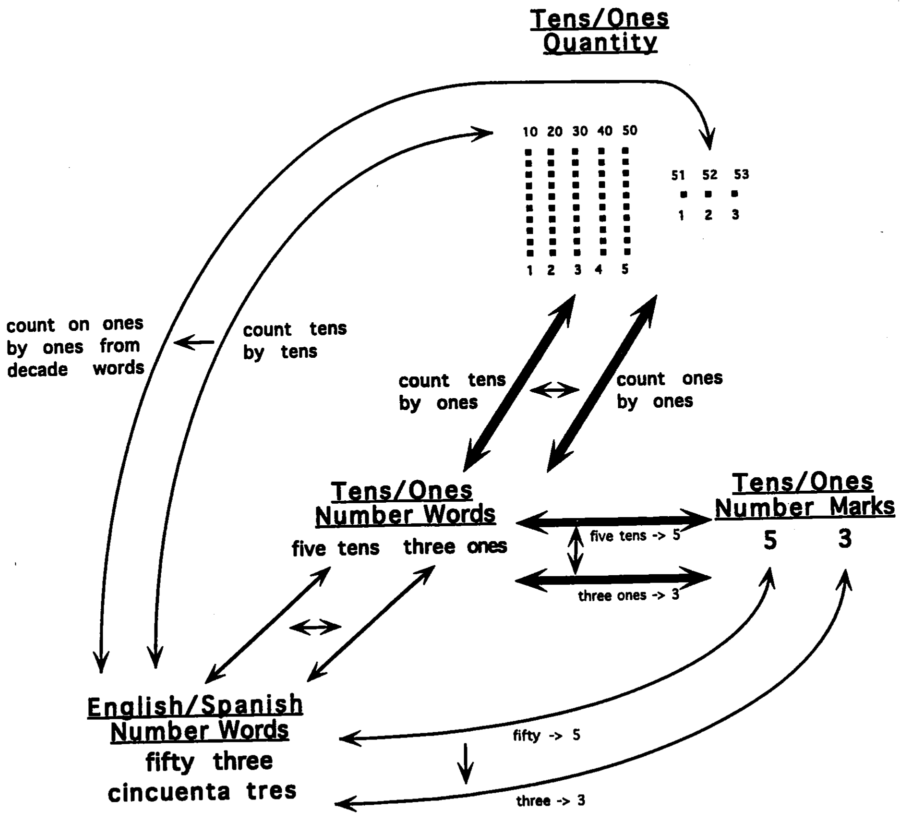


FIG. 8. Sequence and tens/ones quantity counting/making/taking shifts between tens and ones.

three"). With the sequence English/Spanish number words, the child sees ten ones within each group and counts these ten ones by tens: "Ten (ones in that group), twenty (ones so far in all), thirty (ones so far), forty (ones so far), fifty (ones in all)." The child then shifts to the ones quantities, sees each single quantity, and counts on by ones from the tens-count total ("fifty-one, fifty-two, fifty-three"). These strategies thus vary in how the child is seeing/thinking about the quantity and in the counting skills required; counting by tens is required for the sequence method, and only counting by ones is required for the tens/ones method. With both conceptions, the child who is making or taking quantities must remember the number s/he is making or taking (see Fig. 7: remember/monitor/stop at *x*). Also, with both conceptions, children must shift from the tens quantities/words/mark to the ones quantities/words/mark. As we shall see, the shifts are difficult for some children to negotiate.

Each line in Fig 7, except headings and the *if* statements, is a potential stopping or error point requiring tutorial support. Across the three in-depth case studies



(Beschorner, Sartini, Taniguchi, 1995), at least one tutored child made an error at each possible point. No child made all possible errors. The present case study focuses on difficulties in shifting from tens to ones, but some other potential error points in Fig. 7 will also appear in the examples and be discussed there.

## THEORETICAL CONTEXT

Our theoretical context for the tutoring stemmed from our practical goals for inner-city school mathematics learning. We were trying to develop methods that could maximise learning for all children, including the lowest-achieving children, while using the least costly resources possible, because large inner-city school systems are resource-poor. Adults in our project had successfully tutored children before, sometimes helping children who had missed weeks or months of classes function within a few sessions (e.g. Fuson et al., submitted). However, adult tutors may be a scarce resource, whereas high-achieving children exist in every classroom. Therefore we sought to find out whether such children could learn to tutor sufficiently well to help their lower-achieving classmates function successfully on tasks that were being done by the whole class. Otherwise, this potential classroom learning time is lost, and children may even learn and practise errors if they cannot function minimally in the classroom task.

We began with individual sessions with a child tutor and child tutee. If this proved successful and we felt we had found effective general and specific tutoring methods for the domain, our ultimate goal was for the teacher in a classroom to teach to the whole class the general tutoring approach (e.g. don't just tell the answer or do the problem, ask questions, try to help the child do it his/her way) and any specific tutoring learning methods (e.g. counting by tens to one hundred raising one finger for each ten to show how many tens in that decade word). With the teacher's help and reflection, children could then practise various aspects of the tutoring approach, so that many children could potentially become tutors, as least for some parts or some tasks.

Our previous adult tutoring of children was of two main modes: (1) begin with unitary conceptions and move the child at the child's pace through activities to build up conceptions culminating in the most advanced classroom task, and (2) work through the current classroom task, helping the child use current knowledge or build missing knowledge, taking detours into longer learning chunks as necessary. In both types of tutoring, the tutor must simultaneously monitor the overall task and progress through it, while adapting help at a given moment to the interpreted needs of the tutee. Monitoring progress through a sequence of tasks, and learning this sequence, seemed much more difficult than just monitoring the current classroom task, with which the child tutor would be quite familiar, having just been doing it in class. We therefore used the second mode, wondering whether the child tutors could adapt their knowledge of the classroom task to observed difficulties of the tutee.

The nature of the learning domain also influenced our choice of theoretical context. As discussed above, multi-digit subtraction requires a considerable amount of complex cultural semiotic learning. Children must learn the cultural conventions of word meaning, word order, digit meaning, digit left–right position, the complex web of relations that connect these, and lists of number words. There is a great deal of social–arbitrary knowledge here, to use Piaget's term (e.g. Piaget, 1970). This is not a domain that children can construct alone by rearranging pebbles. They must have opportunities to learn and use the conventional quantitative meanings of the words and two-digit written numbers. This, of course, suggests Vygotsky (1934/1962, 1934/1986, 1978), who focused on issues of children's construction of culturally important and heavily culturally infused systems of knowledge as supported by cultural members more expert in that knowledge. A further aspect of this domain also makes Vygotsky a sensible choice: This is that many children's errors and stopping points in the multi-digit addition or subtraction domain seem to stem as much from a failure of the co-ordination and use of what the child already knows or can manifest in some way at other times, as from a total lack of such knowledge. Thus, a major function of the more expert member is to help the learner utilise and coordinate at the necessary times competencies and knowledge the learner may have. Because some of the learning difficulties in this domain, such as aspects of the tens-to-ones shift identified in this case study, seem to result from an uninhibited momentum of particular actions (e.g. counting by tens), the later focus of Luria and other students of Vygotsky on self-regulating speech, and on its initiating and inhibitory effects (e.g. the review in Fuson, 1979; Luria, 1969), is also pertinent.

Our general theoretical context, therefore, is the broad sweep of Vygotsky's perspective on the learner/teacher pair co-constructing a culturally important activity by mutual adaptations in carrying out this activity and by gradual withdrawal over time by the teacher as the learner is able to take over more of the activity. The semiotic constituents of the activity (e.g. mathematical words and objects, ordinary words) possess their cumulated historical–cultural potential for directing and constraining meanings (e.g. the ellipses in the decade words twenty, thirty, and fifty that make their links to two, three, five, and ten less clear) as well as the particular meanings built or used at a given moment by each participant. This general context stems from our readings of Vygotsky (1934/1962, 1934/1986, 1978), the subsequent literature on scaffolding, teaching as assisted performance, and apprenticeship (e.g. Bruner, 1986; Collins, Brown, & Holum, 1991; Newman, Griffin, & Cole, 1989; Rogoff, 1990; Rogoff & Wertsch, 1984; Tharp & Gallimore, 1988; Wertsch, 1985), and more recent work seeking to extend these perspectives (e.g. Forman, Minick, & Stone, 1993; Resnick, Levine, & Teasley, 1991; Sinha, 1988).

We are not focusing in this paper on an analysis of the tutorial interactions, but two particular constructs of Vygotsky are exemplified in the examples we have chosen. First, the learner's progress in these tutoring sessions reflects several aspects of the move from the inter-psychological plane of functioning to the intra-psychological plane, as discussed by Vygotsky (1934/1962, 1934/1986). Both the adult tutors and the child tutors must and did initially carry the whole task and prerequisite competencies in mind and help the learning child, Carolina, build a coherent map of the whole task. They also frequently directed Carolina's attention to crucial features of the problem-solving environment, helped her remember and use relevant knowledge, and helped her carry out problem-solving steps by doing part of the step and then pulling out when she seemed able to continue on her own (or in some cases the learner took over the whole step herself). Carolina gradually became able to direct her own attention to the crucial features, remembered and used relevant knowledge, and carried out previously difficult problem steps. Her subtraction performance became more correct, rapid, and automatic, and eventually was accessible to verbal description and anticipation. Carolina moved, over time, from needing considerable support within the inter-psychological plane to needing less support and finally to independent performance—that is, to the intra-psychological plane.



Second, both child tutors and the adult tutor adapted their tutoring to their interpretation of Carolina's learning zone—what Vygotsky (1978) called "the zone of proximal development". This zone includes all activities within which a child cannot function alone but can do so with the help of another. This construct assumes that a child moves within this learning zone by using initial available conceptions and then by constructing more advanced conceptions and competences-in-use. Thus, a helper can be more effective if s/he has a conception of the child (the learner) as (1) having conceptions of the task and (2) as having conceptions that may differ from those of the helper. Whether our first-grade tutors could conceptualise their tutees in either of these ways was not clear to us when we began. However, several examples of child tutor behaviour in the examples given here do indicate to us that the child tutors did understand that the tutee sometimes had conceptions that differed from theirs and, furthermore, that they used specific conceptions of such a differing tutee conception to invent tutoring moves adapted to a tutee's conception. These are discussed as they occur in the examples.

Finally, there is a very specific way in which the tutoring became adapted to what Carolina could not do on her own—that is, to what she needed to complete her problem solving in this domain. Tutoring in the mode of posing a problem and then helping only where a child cannot carry through independently empirically tests the constraining demands of that situation for that person. Most of Carolina's troubles in this domain are generated by mappings between multi-unit number words, marks, and quantities that are still under construction. The tutor response to Carolina's revealed needs for help gave a particular focus to the interactional discourse: It is not simply on a plane of words or quantities or written marks, but mainly focused by tight mappings between them. The character of the discourse is dominated by tutor eliciting of Carolina's use of such tight mappings between quantities, words, and marks.

The tutoring also posed an interesting test for what the child tutors might be able to do. The task of eliciting Carolina's own mapping from one mode to another, without simply doing it for her or modelling it, make extensive and detailed demands on the tutor's knowledge of the domain and on recognizing and using alternative presentations of this knowledge in the world. We had no idea how much of that the child tutors could do, and part of the motivation for this study was to examine this question. Although this report focuses on Carolina rather than on her child tutors, some indication is given here of the child tutor's general ability to devise scaffoldings for Carolina's errors in such mappings.

## Method

### *Sample*

Carolina had less ability to function independently in a multi-unit activity than any other child in the class (except for two children who entered the school long after basic multi-unit instruction had been introduced, one of them just days before these tutoring studies began in June). She was also among the least advanced in other areas of mathematics (e.g. small-number word problems) and in reading. She rarely handed in any homework. She was very sociable, and both her child tutors were delighted at the prospect of working with her. Just days before this tutorial study began, Carolina started to make progress in reading, her homework started coming in, and, on her own initiative, she taught herself to count to 100 by practising with her sister at home. So, in the examples below, we can follow a child who has a large web of connections yet to learn but is begining to make quite determined efforts to do so.

Two of the top six children in the class were selected as tutors for Carolina because they seemed to have the conceptual knowledge necessary to serve as tutors. Both girls were used as tutors because of scheduling issues, and both functioned quite ably as tutors, providing affective as well as conceptual support adapted to Carolina's needs. They also occasionally helped when help did not seem necessary or missed a helping opportunity, but both also did quite autonomous and helpful tutoring that differed from what the adult tutor had modelled.

### *Tutoring Context*

Over a 17-day period, Carolina and one of her tutors worked together 11 times in a small room adjacent to the classroom, where they were videotaped while they worked at a table. The content of the sessions initially followed the



activities in the classroom. Near the end, some new activities were explored with money, and these were not done in the classroom.

Activities were done in an explicit teacher/student mode, with the stated aim of the "teacher" monitoring and helping the "student". The children initiated switching roles, and the tutors enjoyed being the "student". They changed roles after each one or two problems. The child tutor, as "student", could provide a model of an accurate solution, and as Carolina became better, there were occasional opportunities for her to correct the child tutor. At the beginning of many sessions, Carolina had an opportunity to choose whether to be the student or the teacher, and she always chose to be the student; she seemed quite comfortable in that role.

The adult tutor initially described this teacher/student situation and good ways of helping (e.g. don't just tell the answer, ask your student to count out loud, ask your student to explain an answer) and then helped the child tutor carry out good ways of helping in initial sessions. Particular ways to help specific difficulties were shown for a few difficulties. After the first few sessions, the child tutor did most of the tutoring, with the adult tutor monitoring both children and occasionally helping the child tutor to tutor better or directly tutoring Carolina himself. The adult tutor was very experienced in tutoring children in this conceptual domain.

### *Analysis*

All of Carolina's tutoring sessions were transcribed. All mathematical and tutoring conversations were transcribed verbatim. The few general discussions (e.g. a discussion of playing games carried out while arranging groups of pennies into stacks) were described briefly. Object situations and actions were described. Non-verbal behavior was described (e.g. *Carolina looked at the child tutor, the girls smiled at each other*).

The multi-digit problem component analysis was developed by the two authors and by two undergraduate students who worked on the other two case studies. The first author read all of Carolina's transcripts, watched some tapes, and read all the transcripts of one of the other cases. The second author was the adult tutor for all three case studies and had transcribed part of Carolina's sessions. Each undergraduate student was the transcriber of one of the other two case studies. The problem component analysis thus reflected strategies of all three case students and all three tutors. It is intended to be a general description of object quantity solutions of two-digit take-away situations and is consistent with both authors' observations of many children's two-digit subtraction with quantities.

The first author made notes throughout Carolina's transcript concerning the nature of Carolina's learning difficulties and the characteristics of tutoring. Carolina's persistent difficulties with shifting from tens to ones were the most striking aspect of her case. Examples were chosen to show the major aspects of these difficulties: their reappearance in each new context, their reappearance in a previously successful context, and the gradual learning across time as reflected in the decreased amount of tutor support necessary for success. The range and nature of these difficulties indicate the complexity of the learning task facing children and show the existence of all of the paths shown in Figs. 7 and 8. A few other difficulties demonstrated by Carolina were also included to provide a general context for her tens/ones difficulties.

The examples chosen were numbered consecutively, except for the final two examples, which concern issues other than the tens/ones shifts. Each example was constructed by including the pertinent part of a transcript and then titling and describing that part so that its context would be clear. Interpretive comments were made throughout the example; these are included in brackets to differentiate them from the descriptions of actions or object context that were in the original transcripts in parentheses.

### *Presentation*

Presentation of case studies is difficult: The sheer mass of data for the case study cannot be presented, and consequently, it is hard for the reader to get "inside" the case sufficiently to understand it to any depth or to judge the accuracy of the interpretations of the data by the case study author(s). Our solution to these difficulties is to present excerpts of the actual case in sufficient detail for the reader at least to begin to understand the learning difficulties in their situated contexts and to include our descriptions/interpretations of these difficulties within the case material for the reader to judge the accuracy of our annotations. We will therefore rely heavily on these examples in the presentation of the actual case study and will assume that the reader has read through each example before it is discussed.

## The Case Study of Carolina

### *Overview*

*Carolina's Errors.* Tens-to-ones shift errors can appear as a failure to shift from the tens digit to the ones digit (e.g. for 73 to shift to ones objects and counting by ones but count 7 not 3), from tens objects to ones objects (e.g. say "forty-one, forty-two" but continue to count penny strips rather than pennies), from counting by tens to counting on by ones (e.g. after 4 penny-strips count the 6 pennies but say "50, 60, 70, . . ." rather than "41, 42, 43, . . ."), or from counting the tens to counting the ones (e.g. after 4 penny-strips count the 6 pennies as "5, 6, 7, 8, . . ." rather than as "1, 2, 3, 4, . . ."). Errors can also involve more than one of these failures at the same time.

The major aspects of Carolina's difficulties in shifting from tens to ones are manifested in the examples as follows. Examples 1 and 7 illustrate a failure to shift from the tens digit; such errors are infrequent for Carolina. Examples 2, 3, 5, 6, and 8 concern ten/one shift errors in making the START (see Fig. 7) quantity. Examples 2, 5 and 8 show five failures to shift from tens objects to ones objects. Carolina does learn to make such shifts with penny-strips (the objects in Example 2): Examples 5 and 8 show the reappearance of the object shift difficulty for new objects (sticks and dots and base-ten blocks); Example 8 shows progress in that Carolina easily corrects and smiles at the adult's feedback, as she recognises this old error. Examples 3 and 5 show two failures to shift from tens words (i.e. the count by tens is continued); the second case is a failure to shift both objects and words. Example 6 shows the reappearance of the shift difficulty for penny strips but at a new level of competence: Carolina does not continue on with an incorrect tens count or object; instead, she stops at the proper time but cannot make the shift unaided. Examples 2, 4, 6, and 9 show ten/one shift problems in counting the final REST penny-strips. In Example 2, Carolina continues to count by tens even when she is trying not to do so (i.e. her knowledge of the shift exceeds her capacity to use this knowledge). Example 4 shows progress in that Carolina does shift the count by tens, but she does so one object too late. Further progress is made in Examples 6 and 9, where she stops rather than making an error: These both involve an answer in the teens, so they may also reflect special difficulties with shifting after ten (all other shifts have a pattern of *x*ty, *x*ty-one). In Example 8, Carolina also shows progress in counting the REST quantity by successfully shifting when counting a base-ten–blocks answer for the first time; however, her old error of failing to count the original one does reappear here. Examples 10 through 13 concern ten/one shift difficulties in the new money context, where two kinds of tens must be counted: dimes and stacks of ten pennies.

*Tutor Functioning in Carolina's Learning Zone and from Carolina's Perspective.* Examples 1 (the second half) and 11 demonstrate the child tutor's awareness of and concern with Carolina's conceptions of the activity. In Example 1, Carolina does not know how to break up a ten-stick to take ones from it. The child tutor does not just demonstrate her own method (circling part of the ten-stick and making dots for the remaining ones). Instead, she begins from Carolina's point of view—as possibly not even seeing any ones within the ten-stick—and first makes a column of 10 dots so that Carolina can see these ones dots. The class had not done this for several months, but this is how they had begun to build up ten-sticks. The child tutor then elicits a method from Carolina to show in her own way how to take away 9 of the 10 dots. Then the child tutor explicitly links these 10 dots to a ten-stick by drawing a line through the ten dots to make it look like a ten-stick (though with visible ones inside it) and discusses how they are the same. She then has Carolina try to use this method with a ten-stick. In Example 11, the child tutor thinks of a way to check (and clarify) Carolina's understanding of the equivalence of a dime and a stack of ten pennies by seeing whether Carolina would trade with her. In Example 6, the child tutor views the penny-strip situation from Carolina's point of view and anticipates her possible error of continuing to count tens objects; she removes the extra tens strips from the working space to increase the chance that Carolina would make the shift correctly. All tutors throughout the examples work with Carolina's choice of counting by sequence words rather than trying to impose the child tutor's method of tens/ones words.

In the later examples, tutors withdraw the amount of help more frequently by simply giving feedback when Carolina makes an error rather than giving more information or helping. Such adjusting of the helping level is also done by the tutors within one helping incident. In Example 5, the adult tutor helps Carolina to make the transition to making dots rather than ten-sticks by counting with Carolina while she makes dots for 81, 82, 83, 84 (Carolina therefore can concentrate on making dots) and then stops counting and lets Carolina take over the whole process of counting while making dots for 85 through 89. In Example 6, the child tutor pulls in a ones object (penny) when Carolina fails to do so because she is concentrating on the word shift from 70 to 71. The child tutor continues to pull in pennies for Carolina's counts of 72 and 73 and then stops as Carolina takes over pulling in pennies herself.

### Difficulties in Shifting from Tens to Ones

Example 1 was chosen to show the referential and conceptual complexities in a tutee making, and a tutor supporting, the shift from tens to ones even just within the written number marks and spoken number words. The child tutor spontaneously chooses an appropriate method, framing the two digits with the number words "tens" and "ones". However, pointing to a digit within 79 indicates that digit explicitly and only implicitly indicates its location (and therefore its multi-unit quantity) because position is less salient than the digit itself. Therefore, the meaning referent for the point is ambiguous. The word "one" also has a general referential indicating meaning ("that one", meaning "that thing"). The child tutor uses this indicating meaning of "one" for both the tens and ones positions, thus potentially misdirecting the quantity conception of the tens position as "ones." Carolina negotiates most of these ambiguities with increased tutor support. But the double shifts involved in each case (shifting the digit from 7 to 9 and shifting the quantity label from tens to ones) continue to pose problems, as she makes one but not both shifts. This is a general pattern that will continue throughout her difficulties and will apply to quantities as well. When Carolina attempts to include both the new digit (9) and the new position/quantity label (ones), she says "ninety-one" instead of "nine ones". This demonstrates that the sequence words as well as the tens and ones words are continually potentially activated and in use; here their structure leads to a wrong answer as the ones digit (9) and the ones quantity label (one) fill the sequence slot structure "*x*ty-*y*" and become "9ty-one". In this example, the child tutor



seems to react to the complexity of this task for Carolina and the tentativeness of Carolina's final answer ("one?") by putting together the two parts of the answer elicited from Carolina ("nine" and "ones?" as "Yeah, so it's nine ones") rather than persevering with supporting Carolina to saying a full integrated statement.

This messy and problematic communicative interaction is followed by the child tutor's invention of a way to help Carolina take ones from a ten-stick that shows considerable awareness of conceptual difficulties Carolina might have in understanding this step. This method (discussed in the overview) was extremely successful, and Carolina rarely had problems taking ones from a ten-stick again.

The beginning of Example 2 indicates the independence of sections A and B in Fig. 7: Carolina has for several problems been successfully taking away two-digit quantities (section B in Fig. 7), but here she says she does not know how to make a two-digit quantity (section A). Carolina has been using her knowledge of how many tens in a decade word (e.g. fifty is 5 penny-strips) to take away penny-strips; now, when asked, she uses that same knowledge to make fifty with penny-strips. Carolina responds successfully to the tutor's question requiring a shift to the ones digit (she says "three") and to the ones label (though with a question in her voice: "3 ones?"), but she does not shift to the ones quantities (she continues to pull out penny-strips of ten pennies, not single pennies). She does make the shift to ones quantities under tutor questioning concerning the value of the strips. Carolina exhibits a tens-to-ones shift problem again when counting the final-answer quantities, continuing counting by tens when pointing to single pennies. She has had this problem several times before. This example shows a later point, where she is noticing the problem and trying to self-correct but takes three tries to do so. On this problem Carolina also remembers to count both sources of ones: the ones left on the ten-strip and the original ones. On several previous problems she had not counted the original ones. She closes this example with a statement of confidence and a practice/demonstration of the correct shift, both fairly impressive meta-cognitive actions for a learner with as many difficulties as she has been having.

In Example 3, later that same day, Carolina does not make her earlier error of continuing to count the ones by tens (perhaps because she is already at ninety), but she does count the ones by tens beginning again at ten. Simple feedback that this counting is wrong ("Nah ah") enables her to correct herself ("Oops. 10–90, 91–99."), ending with a mutually affirming eye contact between the child tutor and tutee. In the next problem (Example 4), Carolina makes the shift in counting by tens to counting by ones one object too late (she counts the first penny as "thirty" rather than as "twenty-one"), but she does make the shift correctly without help on the next try. In Example 5, Carolina, in drawing the start quantity 89 with ten-sticks and dots, first shifts neither the counting by tens nor the objects (she continues to count by tens "ninety" and make ten-sticks when beginning to make the 9 ones). She then makes the counting shift in words correctly (begins to count on by ones "eighty-one") but does not shift the objects (draws a ten-stick, not a dot). The adult specifies the quantity to draw ("dots") and the count word ("eighty-one"). Carolina then finishes counting and making the rest of the dots by herself.

In the next session, three days later (Example 6), Carolina, using penny-strips, again fails to make the shift from tens to ones in counting and in the quantities. The child tutor removes the extra ten-strips so Carolina cannot use them and supports with a question Carolina's shift to counting on by ones (to saying "seventy-one"). The child tutor then pulls in pennies (chooses the correct quantity) for the first three ones counts, and then Carolina takes over the whole counting, pulling in pennies herself. Carolina makes the shift to counting on ones for the answer with a prompt from the child tutor, and also corrects her failure to count the original pennies after a simple negative prompt ("Noo.").

Later that same session in Example 7, Carolina makes all of the shifts correctly when making a quantity in ten-sticks and dots except that she perseverates on the tens digit 7 and says it rather than the ones digit 6, though she draws the correct number of dots.

On the next day, the girls ask to use a new kind of quantity, base-ten blocks, that they see sitting in the interview room (Example 8). The blocks had not been used in the classroom and were completely new to the girls. Carolina uses these new quantities almost correctly, but she does not shift from using tens quantities (the long, ten-connected-units blocks) to using ones quantities. However, she needs only a question from the adult to do so, and she smiles when she responds, recognising her old error but evidently feeling enough in control of it to smile at it. She then starts to make the classic error that children make when they do not have enough ones to take away: they add in just enough ones to be able to take away (i.e. they change the original problem). When faced with this issue in question form ("How would you take away 5 ones?"), she generalises use of the yellow covering strip from the penny-strips, which look quite different from the base-ten blocks. In counting her answer, Carolina successfully makes all the necessary tens-to-ones shifts within a sequence count. However, in making the start quantity when doing the problem over again, she shifts within the sequence count ("eighty, eighty-one") but continues to use tens objects. Thus, she can now sometimes, but not yet consistently, carry out the complete shift.

Two days later, in Example 9, Carolina has difficulty making the shift in sequence counting at 10, perhaps because the teens are different from the "*x*ty, *x*ty-one" pattern she has been using for most problems. All the tutored children had a similar special problem with the 10, 11 shift (Beschorner, Sartini, & Taniguchi, 1995). Referential issues are illustrated again. The child tutor's first question is ambiguous ("What comes after ten?" can refer to a count by ones or a count by tens), and both girls use the indicative "one" for the one dot in 51 that Carolina forgot to count (though here it may also have a cardinal or multi-unit meaning).



The next four examples deal with a new set of conceptual difficulties as dimes (US 10¢ coins) are added to the penny-strip tasks: A dime is a single object smaller than a penny, but one dime has the same value as a group of ten pennies, a stack of ten pennies, or the ten pennies on one penny-strip. When sequence counting, each dime must be counted by tens even though it is only one object. Thus, a whole new level of perceptual/conceptual conflict is added to the tens/ones shift. In Example 10, Carolina is pulled by the misleading perceptual cues several times. She counts the 9 pennies as a ten, counts dimes as ones, and counts three dimes as a ten. Finally, on her sixth attempt, she successfully counts stacks of ten pennies and single dimes by tens and counts each of the 9 loose pennies by ones.

Two days later, just before Example 11, it takes Carolina a long time and repeated errors to make 3 single dimes equivalent to 3 stacks of ten pennies. She focuses on the number of objects instead of their value and keeps making 3 stacks of ten dimes. In Example 11, Carolina shows increasingly flexible quantity knowledge as she counts the 3 single dimes by tens, the 3 stacks of ten pennies by tens and also by ones, says several times that the 3 single dimes are the same as the 3 stacks of ten pennies, counts the dimes by tens and then counts on the stacks of pennies by tens, and counts alternating dimes and stacks of ten pennies by tens. In between these successful counting tasks in which quantity overcomes conflicting perceptual evidence (e.g. the oneness of a dime, the oneness of a stack of ten pennies, and the oneness of a penny), she still gives in to perceptual quantity three times. Twice she counts the ten stacks of pennies by ones, perhaps to preserve the usual pattern (in counting the remaining number) of counting on ones after counting by tens, and once she counts the dimes by ones. Near the beginning of the example, the child tutor again uses the word "one" in a mixed way, as an indicative ("stack" could have been used) but also perhaps as a quantity (one stack). This use immediately precedes (and thus may have contributed to) the first counting of the pennies by ones.

At the beginning of Example 12, Carolina has taken from her $1.00 (made from 4 stacks of ten pennies and 6 dimes) 2 dimes and 8 pennies (taken from one stack of ten pennies). She has remaining a row of 4 dimes, then the stack of 2 pennies left from taking 8, and 3 stacks of ten pennies. In counting these, she is faced with the choice of counting the 3 stacks of ten pennies out of order (skipping over the 2 pennies) so she can first do all the counting by tens, or counting the objects in order and therefore counting on by tens from 42 (52, 62, 72), a skill she does not have. She has not faced this problem before. The adult suggests the former strategy, and Carolina successfully counts everything. She counts the dimes by tens, continues to count the stacks of ten pennies by tens, and successfully shifts to ones words and ones objects to count on the two pennies. Carolina then closes with an accurate description of her taking away 28¢ by identifying the tens digit (2) and the dimes as tens and saying she took away 2 and then identifying the other digit (8) as not a ten and so she took away

8 pennies for that (not a ten) digit. Thus, she uses here in her explanation the frame of tens and not a ten (ones) given by the child tutor in Example 1.

Example 13 shows several pieces of the final tutoring session with Carolina (a teachers' strike and Board of Education lock-out in the autumn had led to school running unusually late in June). The beginning segment shows the affective context of the tutoring, with both tutors providing affective support and encouragement. (These were crucial attributes throughout the tutoring sessions, we believe, because the sessions were conceptually demanding of Carolina, though the tutors all tried to stay comfortably within Carolina's learning zone.) In the next segment, Carolina gives an impressive explanation of why she counted her dollar only by tens by giving a hypothetical "regular" quantity that would require the usual counting by tens and ones and then showing that her quantity was not like such a "regular" quantity because it had no single pennies but only tens. Carolina then copies the child tutor's organising of groups of ten pennies by making stacks, but reveals her understanding of a good reason to do so by anticipating her next taking-away action and describing how stacks will enable her to take one stack up and another stack up (as she counts by tens, saying one word and picking up one stack). Carolina still verifies by a questioning tone that a dime is ten before she takes away, indicating some remaining effect of the perceptual oneness of a dime. Carolina then "flexes her intellectual muscles" at the child tutor by telling her "don't be copying" after the adult verifies that Carolina's answer is right. Carolina does not remember from the previous day the strategy of counting all the tens first even if the ones are in the middle. Instead, she shifts to counting all of the pennies by ones when she hits the stack of 7 pennies. When the adult suggests counting all of the tens first, she successfully does so, in marked contrast to her many failures on the previous day. On the next problem Carolina indicates that she knows that her drawing of 10 ten-sticks is equivalent to drawing 100 dots by saying she had done the latter when she had actually done the former. Finally, we end with a correct shift from tens to ones in counting the answer in ten-sticks and dots. So, on this final day, she has successfully on the first attempt shifted from tens to ones both in a mixed dime and stack-of-ten-pennies stack context and in ten-sticks and dots.

### *Early Difficulties in Estimating*

In Example 14 (early in the tutoring), neither child clearly estimates $1 – 99¢ by thinking of the 99 as a sequence word and the $1 as 100¢ and counting up one. The child tutor first gives an answer of 1, which might be a sequence response. But her other responses suggest that she is focusing on the tens and ones and this answer is instead a focus on the ones (i.e. there will be 1 left after taking 9 ones from a 10 in the ten tens making $1). She then focuses on the tens (there will be 1 ten left after taking 9 tens) and then adds these differences to get 11. She does not think about the fact that she took the 9 ones from the tenth ten. This is very early in her use of sticks and dots for such problems; later she almost certainly would have anticipated the whole problem correctly even with a tens and ones strategy. Carolina gives a "wild guess" of 43, indicating little quantity notion of any of the 3 two-digit quantities in the problem.



### *The Child Tutor's Use of Her Multi-unit Web*

Our final example (Example 15) was chosen to illustrate the level of the child tutor's conceptual knowledge in solving a multi-digit subtraction problem. Here, the child tutor is being the tutee. She ordinarily counts the tens using tens/ones words (i.e. counts the tens by ones), but Carolina asks her to do a sequence solution ("Go ten"). The child tutor does so, but uses a combination strategy of counting the ones separately beginning with "one" (as she ordinarily does in her tens/ones counting) rather than counting on by ones from twenty as in the usual sequence strategy. Because she has taken the 5 ones from a ten rather than from the available 9 ones, she gets a total of 14, which she then needs to add to the 20 (the ordinary sequence strategy of counting on the ones from the tens eliminates this need because she would just have counted from twenty to thirty-four). She solves this problem by making a mental ten while counting the ones again ("That's ten; that's one ten") and then counting the remaining ones ("One, two, three, four"). She then mentally adds the new one ten to her sequence count of tens (gets "thirty") and counts on the four remaining ones (which are isolated from the ones making her ten only in her own mind) to get the total 34. She then describes quite clearly what she has done. She thus demonstrates a strong ability to negotiate within the full web of multi-digit knowledge, moving from tens/ones words and strategies to sequence words and strategies and mentally composing and adding tens within either conception. This flexible and fluid performance is the goal of our classroom and tutoring interventions.

## CLOSING COMMENTS

Carolina's errors demonstrate the many complex pieces of the multi-unit web of knowledge that must be learned and deployed in concert. Ten-structured counting methods, whether using sequence or tens/ones words, require a shift from counting the groups of ten to counting single entities. In sequence counting to make a quantity (Carolina's favoured method) this tens/ones counting shift requires coordinating the following shifts: (a) visual gaze from the left mark position to the right mark position, (b) the short-term memory number being monitored from the digit in the left position to the digit in the right position, (c) visual gaze from the last group of ten objects to the ones objects, (d) counting path planning from the groups of ten objects to the ones objects, (e) counting list used from counting by tens to counting by ones, (f) counting on from the last ten counted, not counting beginning from one. If the counting is done by remembering the whole sequence word rather than looking at the written two-digit marks, (a) and (b) are replaced by shifts (g) from the decade part of the sequence word to the ones part and (h) the short-term memory number being monitored from the number of tens in the decade word to the number of ones.

Each of these pieces could and did drop out for Carolina, but shifting the kind of object and shifting the counting by tens to counting on by ones were by far the most frequent errors. There seemed to be a natural momentum in the counting activity that maintained both the nature of the count (the ten count) and the type of objects counted (groups of tens). It took considerable effort and attentional/conceptual resources for Carolina to begin to make this shift, and then the momentum still seemed to propel her into errors for quite some time after she first began to make the shift successfully. This shift needed to be learned and used with each kind of pedagogical object. For several sessions, any change of pedagogical object elicited an error again, even if Carolina had earlier made the shift successfully with that object.

Carolina showed the most competence with taking away tens and ones; making the start quantity and counting the final answer both contained more errors. Carolina did have an opportunity to learn the taking-away component in class, where for several days the children made $1 with ten penny-strips and took away various amounts. The main difficulty that Carolina had with the taking-away component quickly disappeared (Example 1). At first, she did not know how to take ones from a stick ten; the ones are not displayed in a ten-stick as they are in a penny-strip. But once the child tutor had talked her through circling part of a ten-strip and then counting up to make ten (drawing the dots as she counts on: see Fig. 4), Carolina rarely had problems with this component again. The majority of Carolina's most advanced demonstrations of knowledge concerned this taking-away component of subtraction. She successfully generalised the yellow covering strip to base-ten blocks in the face of considerable perceptual differences between the blocks and penny-strips, she described taking-away actions clearly, and she anticipated taking away stacks of pennies and commented on the utility of stacking her pennies in stacks of ten.

This taking-away component is more difficult than the final counting-of-the-answer component in at least two ways: (1) for the final count of the rest, no memory feedback loop need be established or monitored, and (2) counting actions are much more familiar than are taking-away actions, especially considering the different taking-away actions required for penny-strips, ten-sticks and dots, and stacks of pennies and dimes. However, Carolina used a collected-tens conception for the take-away component, and sequence conceptions for making the start and counting the final answer. Using tens ones words and counting is much easier for children who have troubles with the sequence to 100 or counting by tens (Beschorner, Sartini, & Taniguchi, 1995). Carolina had just learned the sequence, so perhaps it was not yet automatised enough to make stopping easy.

Smith (1994) also found that shifting from tens to ones was relatively difficult for kindergartners. It took kindergartners a mean of about 1 session or less to



establish meaningful tens groupings, count by ones along tens scaffoldings to 100, and link quantity, words, and marks representations of tens. Shifting from a count of tens to a count of ones when joining tens and ones required a mean of 7 sessions. It was hypothesised there that the attentional demands of juggling two different grouping-evaluational systems within a single operational focus might be a key limiting factor. The tutorial help that was given in that study scaffolded the attentional demands of the task, much as they did for Carolina, and over roughly the same number of sessions.

Carolina exhibited two kinds of influences on her learning that are typical of pre-operational children. Misleading perceptual features often overwhelmed her fledgling multi-unit competencies, and she found it difficult to attend to and consider two different aspects of a situation. Tutoring support was necessary to help her to construct and automatise multi-unit competencies that were strong enough to withstand misleading perceptual features such as the singleness of a dime. Such support was also necessary for her to negotiate her shifts in attention from all of the tens aspects of a given situation to all of the ones aspects.

Multi-digit subtraction in which ones must be taken from a ten is not usually taught in the first grade in the United States (Fuson, Stigler, & Bartsch, 1988; Fuson 1992b). In this study we considered only the early strategies for such problems in which the quantities are made with objects. The child tutor in the final example indicates the kinds of mental methods that follow and depend upon such object experiences. Our final project goal is to have most children able to carry out such subtraction only by writing numbers that record and reflect such actions on quantities (i.e. written subtraction with numerals that is quantitatively meaningful); many children in Carolina's class will be able to do so by the end of second grade. However, even the precursor goal of object quantity two-digit subtraction carried out by ten-structured (rather than unitary) methods is unusually accelerated if it is a goal for all children at first grade, as it was here. Neither traditional mathematics instruction nor instruction generally consistent with the reform mathematics standards of the National Council of Teachers of Mathematics (National Council of Teachers of Mathematics, 1989, 1991) place this topic so early. In contrast, Kamii (1989) reported that many second graders in her Piagetian mathematics classroom had considerable difficulty with such subtraction, and she recommended that it be postponed until third grade (and multiplication be done in second grade instead). A case-study child from the Purdue Problem-Centered Learning Project, a constructivist approach, did not begin to solve such problems in ten-structured ways until third grade (Lo, Wheatley, & Smith, 1994), and this late timing seemingly was not a concern of the project.

The range and nature of Carolina's difficulties indicate the complexity of the learning task facing children and show the existence of all of the paths shown in Figs. 7 and 8. This learning task is much more difficult for children speaking European languages because they must learn the extra sequence strategies and link them to tens and ones knowledge. The great difference between Carolina and the child tutors illustrates how big the gap between the highest and lowest children in a class may be. The child tutors were not only able to solve the problems in the domain correctly; their knowledge was sufficiently robust and accessible to be able to be used in novel ways to help Carolina at many different problem points. It is our experience in teaching this domain to first and second graders for three different years that most children need systematic sustained classroom activities over weeks (and for some children, months) in order to construct the elements of the multi-unit web (Smith & Fuson, in preparation a). The top quarter becomes quite competent just from these classroom activities. The next quarter to third becomes able to function in the problem domain but without the overall control and flexibility of the top children. The lower third to half of the class needs extra support methods; these children either are not rapid learners or begin with prerequisite competencies missing (e.g. count to 100 by ones or tens).

This study demonstrates how much a young child can learn with support. Smith (1994) found that understanding two-digit and even larger numbers quantitatively is within the reach even of most kindergarten children if they are given individual adult tutoring. Deciding when in the curriculum to place mathematical topics is a complex social–political decision that can be informed by research. This study indicates that complex topics can be learned with understanding before their usual curricular placement if that learning is supported with pedagogical objects and activities adapted to children's thinking and to the conceptions they must construct and if adequate learning support is given to individuals who experience the most difficulty in learning.

This study also indicates that peer tutors can learn to provide such support in individual settings. This support is adapted to the learner. How well this can generalise to support within the classroom is a question for future research. This is an important practical issue, given the range and number of errors Carolina made in attempting to solve these problems. When working alone in the classroom on such problems, she seemed quite likely to be practising incorrect strategies rather than learning correct ones. Without the tutoring help, she still had many difficulties, even after the weeks of related classroom activities. The very large gap between the highest and lowest children in a class therefore raises difficult issues of equity (whose learning needs should be served?). A compromise position placing topics after the fastest children can learn them but before the lowest can learn without considerable support is the usual resolution of this quandary. However, seldom do schools address sufficiently the practical issues concerning the mobilisation of such support within school settings for the slowest learners. This seems particularly necessary in the complex domain of operations on multi-digit numbers, perhaps especially for children speaking European languages. Providing learning support adapted to the thinking of other children can be an important learning task for high-achieving children,



permitting them to use their web of knowledge in new ways not enabled by ordinary classroom tasks.

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# APPENDIX

## Example 1

6/7 Difficulty moving from CSD (concatenated single-digit) words to tens and ones words drawing ten-sticks and ones; confusion of sequence and tens/ones linguistic structures; child tutor invents way to help Carolina take away ones from a ten-stick

The problem is $1 – 79¢ = ? You have one dollar, you are buying something in a grocery ad (here, it costs 79¢), how much change will you get? Carolina has already solved the same problem with the penny-strips. She did learn to take away 9 using the coverer. Carolina made 10 ten-sticks to show $1, circled 7 to take away 7 tens, and she can't go on. She needs to see/make ones in a ten-stick to take away 9 ones. [$1 – 79¢]

Carolina, child tutor B, adult

Child tutor (points to the 7 in the 79¢ price tag): What's this? [*Number or position ambiguity.*]
Carolina: 7.
Child tutor: Ok, how much is this gonna be (7)? [*Still some ambiguity*] 7 what? [*Disambiguates by using number in question.*]
Carolina: 7.
Child tutor: 7 what? 7 tens or ones? [*Simplifies and further specifies meaning by giving quantity choices.*]
Carolina: 7 tens?

Child tutor: Ok, if that one is tens (pointing to the 7), then what should this (pointing to the 9) be? [*Uses tens and one choice to support correct quantity word.*]
Carolina: 7 ones [*Switches to correct tens/ones label, doesn't switch number from 7 to 9*].
Child tutor: Look again. This one (9), not this one (7). This one (9) [*Use of "one" as an indicator, not as a number or position or quantity*].
Carolina: 9 . . . 91? [*Ninety-one instead of nine ones; trying to put together the 9 and the ones position quantity label but shifted into a sequence word that combines nine and one.*]
Child tutor: No, look again. First of all, how much is here (9)?
Carolina: 9.
Child tutor: Okay. If this one's tens, then which one should this be? [*Again indicator use of "one" pointing to the numbers 7 and 9 but meaning the positions.*]
Carolina: Ones? [*Understanding the positional referent and perhaps supported by the tutor's posing of the explicit choice "tens or ones" nine moves before.*]
Child tutor: Yeah, so it's 9 ones [*States correct number and quantity together*].
(Child tutor then invents a way to show Carolina how to take out 9 ones from a ten-stick by drawing 10 vertical dots on the ten-stick and circling 9 of them.)
Adult: Now, the next question Carolina is figuring out is how does she take out 9 ones. See if you can teach her that, child tutor.
Child tutor: I don't know . . .
Adult: Okay, that's hard to teach?
Child tutor: Wait. First um, like um, wait. . . . Can I have a piece of paper to show her? If you have like nine dots (draws 10) [*starts with the 9 dots Carolina needs to take away but draws them individually as a column of 10 dots so they can be seen as ten ones by Carolina*], and you need to take away 9. How would you take away like this? (To Carolina) [*elicits a drawing method of taking away that is chosen by Carolina*].
Carolina: Circle?
Child tutor: Um hm. Can you do it for me here?
(Carolina circles nine dots.)
Child tutor: And one left, right? So this is the same thing as tens [*she means the stick-tens*], only it's in dots [*the column of 10 dots is not connected*]. So it would be the same if it was like this too (draws a line through the 10 dots to make it look like a ten-stick). And you would still go like that (makes dot to side of line, outside of circle to show dot remaining after nine are taken away). So, you want to do it on that page now? [*Has Carolina practise with the stick ten.*]
Carolina: Yeah.
Child tutor: You want to take away the 9 ones.
(Carolina circles most of the stick ten on her original problem and puts a dot at the top. [*This was never modelled on an undotted ten-stick; she makes this generalisation herself.*])
(Child tutor looks up at adult, smiles, and nods.)
Adult: Got it? (to Carolina)
Child tutor: Yeah, she has.
Adult: Really good. And show us how much you have left, Carolina.
Carolina: 10, 20 (pauses).
Adult: 10, 20 (points to the one dot).
Carolina: 21? [*Makes shift from tens to ones, but tentatively.*]
Adult: Good work, Carolina.
Carolina: Can I be the teacher now?
Adult: Yeah, you can be the teacher.
Child tutor: Ah, good.
(Carolina and child tutor laugh. Carolina "teaches". She makes up a problem and gives it to the child tutor; child tutor solves and explains it quickly and correctly. Carolina watches closely throughout this process.)



## Example 2

6/10 Difficulties shifting from tens to ones in making a penny-strip START quantity and later in sequence words in counting the REST quantity

For several days, children in Maths class have been buying things from adverts with $1. In this problem, they imagine they have only 53¢ (instead of $1) and will buy something costing 25¢. They are using paper strips showing 10 pennies (and 1 dime shows on the back) and loose pennies. [53¢ − 25¢]

Carolina, child tutor A, adult

(Carolina says she doesn't know how to make 53¢ [she has been taking away such quantities].)
Adult to child tutor: When somebody has trouble, sometimes you have to stop giving them problems, and you just have to start to work on the thing they need to know. What Carolina needs to know is, she needs to figure out how to make 53¢. She's got a number there, 53. (Adult models this for child tutor.) Let's see what that number up top is telling us. What is it telling us, Carolina? How many tens is it telling us to make?
Carolina: 53 (fifty-three)? [*She answers the question, "What is that number up top?"*]
Adult: Yeah, so first of all, make 50 (fifty) [*works with the child's choice of sequence word and identifies the first step; knows fifty is five tens so can put out 5 ten-strips*].
Carolina (touches some tens-strips): 50? How do I . . . ? (She counts silently.)
(Adult asks her to show her counting.)
(Carolina counts 1–5 of the ten-penny strips [*shifts correctly from sequence to tens word: fifty to five tens and counts them by ones*].)
Adult: 5, that's 50. (To child tutor) Is she right so far? (Child tutor nods.)
Adult: Yes. What else do you have to make on top?
Carolina: I gotta take 3 of them? [*Shifts to the mark in the ones position.*]
Adult: 3 what? [*Adult supports the quantity meaning of the ones position.*]
Carolina: 3 ones? [*Correctly identifies quantity at verbal level.*]
Adult: Yes.
(Carolina reaches for 3 ten-penny strips [*said ones, but still uses tens objects; does not shift to ones objects*].)
Adult: Are those ones?
Carolina: No.
Adult: Those are what?
Carolina: Tens [*She knows their quantity, just did not shift*].
Adult: Um-mm.
(Carolina pulls in 3 pennies [*knows and chooses correct quantity*].)
Adult: There you go. Now. . . . Okay, Carolina, now what do you do? You've got your. . . . This is the money you've got in your pocket.
Carolina: I've got to take away this (points to bottom number).
Adult: Okay, you do that for us, and we'll watch you. And you can use this (yellow covering strip) to take away ones if you need to.
(Carolina quickly, independently takes away 2 ten-penny strips, covers 5 ones on a penny strip, begins to count what's left [*she knows well the take-away object component for penny strips, having practised this within problems taking from $1*]. She counts the 2 remaining whole ten-penny strips "10, 20," then begins counting the 5 single pennies left on the covered ten-penny strips as "30". Stops, re-counts, but still counts 10–30 [*cannot shift from sequence tens words to ones words even though trying to do so*].)
Adult: Ah are tho . . . (turns to child tutor) What do you say?
(But Carolina self-corrects first.)

Carolina: 10, 20, 21 (adult nods), 22–25 (then counts original loose pennies) 26, 27, 28 (writes answer).
Adult to child tutor: Is she right?
(Child tutor and adult nod.)
Adult to child tutor: Notice Carolina made some progress this time. This time, she didn't forget those (pennies) over there (the original 3) [*the original ones are physically separated from the pennies left uncovered on a ten-strip, so sometimes one group is omitted from the final total count*].
Adult to Carolina: You still sometimes forget those ones. Sometimes you think they're tens. [*Carolina has had difficulty shifting from tens objects to ones objects over several problems.*]
Carolina: I can say it. I think I can't forget it now (she recounts), 10, 20, 21–28 [*demonstrates the shift from tens to ones words and self-practises*].

## Example 3

6/10 Does not shift from counting by tens when counting pennies in the START quantity

The problem is having 99¢ and buying something that costs 59¢. This is solved using penny strips. Carolina is making the 99¢. [99¢ − 59¢]

Carolina, child tutor A, adult

(Carolina counts 10–70, moves yellow covering strip out of the way, starts to count other 2 tens. Child tutor's pencil drops, Carolina looks down, then back, sees 2 pennies and counts the remaining 2 ten-strips as 80, 82 [*intrusion of "two" because saw 2 pennies?*].)
Child tutor: 90 (nods—she is seeing the 9 tens even though Carolina very quietly called them 82). What else? (Points to written 9 ones.) [*Scaffolds shift to finding how many ones.*]
Carolina: Another 9?
(Child tutor nods.)
Carolina: Can I go with these? (pennies) [*Checking that the chosen ones quantities are correct.*]
Child tutor: Yeah.
(Carolina counts 9 pennies, but as 10–90 [*shifts to counting ones quantities but does not shift to counting by ones—still counts by tens*].)
Child tutor (smiles, turns to adult, but adult is writing): Count it out loud.
Carolina: 10, 20, 30.
Child tutor: Nah ah [*feedback that counting is wrong*].
Carolina: Oops.
Carolina (recounts on strips): 10–90, (on pennies) 91–99.
(Child tutor and Carolina make eye contact [*affirming their interaction and mutual success*].)

## Example 4

6/10 Late shift from counting by tens to counting by ones in counting the REST quantity

The problem is having 75¢ and buying something that costs 48¢. The 75¢ has been made using penny-strips. Carolina is ready to take away 48¢. [75¢ − 48¢]

Carolina, child tutor A, adult

Carolina (correctly takes out 4 tens-strips): 4.
(Child tutor tells Carolina to use the yellow covering strip.)
Carolina (covers up 8 ones, counts the remaining 2 ten-strips): 10, 20 (the first remaining one) 30, (next remaining one) 31–35 [*shifts from counting by tens to counting by ones one object too late*].
(Child tutor shakes her head.)
Adult (to child tutor): Ask her to count them again.
Carolina: 10, 20, 21, 22 (then counts five pennies in original 75¢) 23–27 [*makes shift to ones words at first ones object*].



## Example 5

6/10 Difficulty shifting the sequence counting and the drawn quantity from tens to ones in making START quantity with ten-sticks and dots

The problem is having 89¢ and buying something that costs 81¢. This is solved using ten-sticks and dots. [89¢ − 81¢]

Carolina, adult

Carolina (scaffolded by adult, draws 8 ten-sticks counting by tens; but she does not shift to ones quantities [dots] or to counting by ones): 90 (instead of 81, and drew a stick, not a dot).
Adult: Oops.
(Carolina erases the last ten-stick [*so perhaps knows that and where she did not shift correctly*].)
Adult: Eighty . . . [*backing up to last correct step*].
Carolina: One (but draws a ten-stick again) [*shifts the sequence words to ones but not the quantity*].)
Adult: No. 81, 82 . . . say it out loud. Dots [*specifies the ones quantity and the next sequence words when counting by ones*].
(Carolina counts 82–89 as she makes dots [*adult withdraws support when not needed*].)

## Example 6

6/13 Difficulty shifting the sequence counting from tens to ones making START quantity and counting the REST quantity in old (penny strip) medi-um

The problem is having 76¢ and buying a pencil that costs 56¢ solved using penny strips. [76¢ − 56¢]

Carolina, child tutor A, adult

(Carolina has made 10 penny-strips instead of stopping at 7 [*intrusion of earlier problem situations where they begin with $1*]. Child tutor shakes head at Carolina.)
Carolina: I mean, wait, 10–50, (child tutor starts touching pencil coordinated to Carolina's count) 60, 70.
(Child tutor moves the rest of the tens away with her pencil [*Reducing error of making 10 tens*].)
Carolina: 70 (trails off, looks up).
(Child tutor waits, then casually hints with her pencil, tapping it around the pennies, and moving the tens away from the pennies [*suggesting ones quantities to count next*].)
Carolina: Seventy, seventy.
Child tutor: Seventy what? [*Supporting next sequence count word.*] (Child tutor pulls other ten strips further out of the way [*so can't count tens as ones*].)
Carolina: 71 [answers *correct sequence word with ones but doesn't count ones quantities*].
(Child tutor pulls in pennies with pencil for Carolina's counts of 71, 72, 73 [*chooses the correct quantities—ones quantities*].)

Carolina: 72. . . . 73 (looks over at number written down) [*checking how many ones to count*].
(Carolina takes over pulling in pennies herself, counting much more quickly and confidently.)
Carolina: 74, 75, 76 (Carolina then re-counts ones silently to herself to check).
Child tutor: Ok, how much you got to take away? [*Supporting finding TAKEAWAY number.*]
Carolina: Um (looks at paper) 56?
Child tutor: Yeah. Take it.
(Carolina places hands across 5 tens [*shifted from sequence word "fifty" to tens words "five"*.]
Carolina: Can I take away 5?
(Child tutor purposely doesn't respond.)
Carolina (re-counts 4, then re-counts 5 tens, pulls them away): 5 [*counting the tens, not by tens as in her usual method*]. (looks at written number on sheet): I need the yellow thing (the strip to cover pennies).
(Child tutor looks at Carolina.)
Carolina: Wait, I need the yellow thing.
Adult: Here's the yellow thing.
Carolina (counts 6 units on penny-strip; covers 6) [*she could have just taken the 6 pennies in 76¢*]: *I know. Don't tell me what's the answer.*
Carolina: 10 (starts to count units, pause) [*can't make shift to ones*].
Child tutor: What comes after 10? [*child tutor provides an appropriate and helpful prompt*].
Carolina: 11, 12, 13, 14 (the pennies left on the penny-strip after 6 are covered with yellow strip).
(Child tutor is counting loose pennies in the original 76¢ while Carolina is counting on the strip. Carolina looks over at loose pennies left.)
Carolina: I mean 10 (stops, her hand has moved yellow strip, she and child tutor fix it).
Carolina: 10, 11, 12, 13, 14. The answer is 14 (starts to write) [*has forgotten to count loose pennies, which she previously did correctly*].
Child tutor: Noo (looks over at Carolina).
Carolina (looks over at loose pennies): I mean, 14.
(Child tutor starts moving loose pennies up to top of row after Carolina counts them [*to show all of the ones together*].)
Carolina: 15, 16, 17, 18.
(Carolina takes over moving the pennies, making a vertical column of pennies.)
Carolina: 19, 20. 20 (writes answer).

## Example 7

6/13 Perseverance of tens number when making ones dots for START quantity

The problem is having 76¢ and buying a pencil that costs 56¢ solved using ten-sticks and dots. The problem was already done with penny strips. [76¢ – 56¢]

Carolina, child tutor A, adult.

Child tutor: What about the sticks and dots?
Adult: Ohh! I forgot. You are a better teacher than me. We have to do the same problem with sticks and dots. Do the same problem with sticks and dots, Carolina.
(Carolina laughs, starts to do sticks-and-dots problem. She very quickly makes 7 sticks, recounts.)
Carolina: 70?
Child tutor: Yeah.
Carolina: And (quietly) 7 (but silently makes 6 dots) [*incorrect intrusion of tens number for ones number but makes correct ones quantity: says 7 but makes 6*].



## Example 8

6/14 Brief difficulty switching from tens to ones objects making START quantity in new medium (base-ten blocks) but does generalise covering to take ones from a ten

The problem is having 83¢ and buying something that costs 45¢ solved using base-ten blocks (single-unit cubic-centimetre blocks and ten-connected-units blocks). The girls have not used base-ten blocks. The blocks were sitting in the interview room, and the girls asked to use them. [83¢ – 45¢]

Carolina, child tutor B, adult

Carolina (quickly counts out 8 tens, looks at number, looks at blocks): How many . . . (while reaching for tens blocks) [*switched to ones mark but not to ones objects*].
Adult: Are those ones, Carolina?
(Carolina smiles and shakes head, reaches for ones cubes [*recognising her old mistake*].)
Carolina: I need more [*there are only 3 units, and she needs to take away 5 units*].
Adult: How many ones do you need, Carolina?
Carolina: 3 [*misunderstands as "How many ones do you have?"*].
Adult: There we go. (Carolina puts 3 ones out.)
(Carolina takes 4 tens-blocks then 3 ones-blocks out; then starts to count in 5 ones-blocks [*so she can take out 5 ones-blocks: this is a common error with base-ten blocks*].)
Adult: Oh no. Oops. You were about to add ones in. Is this an add problem?
Carolina: No.
Adult: It's a takeaway. (He puts the 3 ones back.) How do you take away the 5 ones, Carolina?
(Carolina thinks, says "Oh", and spontaneously reaches for yellow strip to cover 5 units in one ten-block [*generalises use of yellow strip from the penny strips*]. She covers other ten-blocks in this process because the yellow strip is much bigger than the ten-blocks.)
(Adult suggests she pulls one ten-block down and cover it.)
(Carolina accurately counts 10–50, 51–55—(the five uncovered units on a ten-block [*shifts sequence words and objects from tens to ones accurately in this first time she counts base-ten blocks as an answer but forgets to count the 3 units in the original 83*].)
(Adult points to the 3 units she forgot in the 83.)
(Carolina counts to 58 [*answer should be 38 but 2 ten-blocks got back into the tens during the problem solving so the problem solved was 103 – 45*].)
(Child tutor got 38, so Carolina re-does problem.)
(Carolina tries to make 83: 10, 20, 30, . . . , 80, then continuing to pull in ten-sticks counting 81, 82 [*shifts to sequence ones but again does not shift to ones objects*].)
Adult: Oh?! Are those ones?
(Carolina shakes head and reaches for 3 ones.)

## Example 9

6/15 Difficulty switching from tens to ones in counting REST quantity in ten-sticks and dots

The problem is 51 – 33 in ten-sticks and dots. Carolina has made 51 in ten-sticks and dots and taken away 33. In taking the 3 from a ten-stick she counted on from 3 but made 6 dots instead of 7, so the answer ten-sticks and dots are one too small (17 instead of 18). She has trouble counting the REST quantity. [51¢ – 33¢]

Carolina, child tutor A, adult

Carolina: Ten (pause) [*shift may be more difficult because the teens don't follow the xty-one pattern Carolina has used on most problems*].
Child tutor: What comes after ten?
Carolina: 20, I mean (long pause) [*the answer to the question depends on whether one is counting by ones or counting by tens*].
Child tutor: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 [*clarifying her meaning of her question*].
Carolina: 11. OK. 10, 11, 12, 13, wait, 10, 11, 12, 13, 14, 15, 16 (writes 16).
Child tutor: Ah ah ah ah ah. How about that one right there? (points) [*Carolina missed the 1 dot in 51*].
Carolina: Oh, that one there (erases; writes 7) [*adds one to the correct position: both girls' uses of "one" had indicative and quantity and cardinal meanings*].
Adult (to child tutor): Is that right?
Carolina (shows the adult what she did): *Ten*, 11–17 [*self-practises shift, as she did in Example 2*].

## Example 10

6/21 Difficulty counting tens and ones quantities in new medium (dimes, stacks of ten pennies, and loose pennies)

The problem is to count 6 stacks of ten pennies, 3 dimes, and 9 loose pennies (99¢); Carolina watches child tutor B count these to get 99¢ and then is supposed to do the same.

Carolina, child tutor B, adult

Carolina: 10, 20, 30, 40, 50 (groups of stacked penny tens, stops when she reaches the dimes, looks at adult) [*does not know what quantities dimes are; conflict between dimes being single entities like ones but having value of ten; also misses one stack of ten pennies*].
(Adult rearranges single pennies to make them more visible to Carolina.)
Carolina: 10, 20, 30, 40, 50, fifty-. . . (then wants to begin counting loose pennies) [*still doesn't know how to deal with the dimes; still missed one stack of pennies*].
Adult: Nope, we're not done with all of the tens.
Carolina: 10, 20, 30, 40, 50, 60 [*counts group of nine pennies as a group of ten, skipping one stack of ten pennies*].
Adult: Nope, those are the ones.
Carolina: Oops! 10, 20, 30, 40, 50? fifty-one, fifty-two [*counting dimes as ones; still skipping one stack of ten pennies*].
Adult: No, that's a ten..
Carolina: Wait. 10, 20, 30, 40, 50, 60, 70? [*First time counts all 6 stacks of ten pennies to 60; counts all three dimes as one group of ten when says 70.*]
Adult: 70 . . . (points to one dime).
Carolina: 70, 80, 90, 91–99 [*on her sixth attempt, counts each dime as ten and shifts successfully to ones words and ones quantities as counts 9 loose pennies*].

## Example 11

6/23 Difficulty in establishing quantitative equivalence of 3 stacks of ten pennies and 3 dimes

Child tutor B has just finished a very long segment of getting Carolina to make 3 dimes as equivalent to 3 stacks of ten pennies; Carolina originally made 3 stacks of ten dimes, using the number of objects rather than the money value.
Carolina, child tutor B, adult



Child tutor (to adult): She got it. (To Carolina): Now, can you tell me what number that is all together? (The three dimes Carolina has just made.)
Carolina: Yeah, 10, 20, 30.
Adult: Right. Okay. And ask her to count that (3 stacks of ten pennies).
Child tutor: Now, count this (points to pennies). Remember, each one is a ten [*here, "one" indicates one stack of ten pennies, a tens quantity, but "one" could erroneously be taken to mean one penny or the value of a penny because the pennies are indicated*].
(Carolina counts by ones 1–31 (should get 30) [*she ignores the stacks of ten*].
Adult: Okay, Carolina, you counted those by ones. Can you count those tens by ten?
Carolina: Yeah. 10, 20, 30 (adult and child tutor count along with her).
Adult: And what's this (dimes)?
Carolina: Thirty.
Adult: 10, 20, 30. Are they the same?
Carolina: Yeah.
Child tutor: Wait, wait, wait. I want to do something. So if it's equal, would you do this? Carolina, would you trade me for these (three dimes)? I have 30 cents, you have 30 cents. Can I trade you? [*Using willingness to trade as a definition and test of equivalence.*]
Carolina: Yeah (they trade).
Adult: Okay. So. Could you count both piles again, Carolina?
Carolina (counts penny piles): 10, 20, 30 (counts dimes) 10, 20, I mean, 1, 2, 3 (doesn't label their quantity) [*still pulled by their single-objectness rather than their monetary value*].
Adult: Three ones?
Carolina: Yeah. I mean no.
Adult: Count them again [*counting 1, 2, 3 and saying 3 tens or 30 would be correct, but Carolina usually sequence-counts*].
Carolina: 10, 20, 30.
(Adult whispers to child tutor.)
Child tutor: How much do you have with all of these (pennies) and all of that (dimes) together, all together?
Carolina: 10, 20, 30 (dimes), 31, 32, 33, 34, 35, 36 . . . [*switches to counting the pennies by ones, perhaps because the tens-to-ones shift almost always occurs when counting objects*].
Child tutor: Wait. Is this a ten (stack of ten pennies)?
Carolina (starts over): 10 . . .
Child tutor: Is this a ten (stack of ten pennies), all of this?
Carolina: Yeah. 10 . . .
Child tutor: So is this (one dime) equal to this (stack of ten pennies)?
Carolina: Yeah.
Child tutor: So this (stack of ten pennies) is like that (one dime).
Carolina: 10, 20, 30, 40?, 50, 60 [*counts the single dimes and stacks of ten ones objects by tens*].
Child tutor: She got it.
Adult: She got it. So let me ask you one more question, Carolina. Could you count it like this? Let's pretend we were counting it like this, in-between (adult arranges piles so a stack of ten pennies alternates with a dime).
Child tutor: That's what I was going to do. That's what I was trying to get at.
Adult: Oh, great. And now, Carolina, count it like this. Now, count them.
Carolina: 10, 20, 30, 40, 50, 60 [*succeeds on this task—maintains value of dimes as ten in the face of perceptual oneness*].

## Example 12

6/23 Success and articulate explanation with sequence tens/ones shifts in counting REST quantity made of dimes, ten stacks of pennies, and loose pennies (large progress all in one long session)

Carolina had many difficulties in this long session with ten/one shifts, but at the end the girls have each made $1 with 4 stacks of ten pennies and 6 dimes. The problem is to take away 28¢ and find how much money is left. ($1 − 28¢]

Carolina, child tutor B, adult

(Carolina has taken away 2 dimes and 8 pennies and has counted them twice. She now counts to find the amount of money left.)
Carolina: 10, 20, 30, 40 (dimes) (moves to the pile of two pennies remaining from a ten-stack when the 8 pennies were taken away; stops, maybe beginning to count 41) [*she has not counted the 3 stacks of ten pennies remaining; has not coordinated the two kinds of tens before counting the ones*].
Adult: Well, let's keep these ones to the side. Count all your tens left first.
Carolina: 10, 20, 30, 40, 50, 60 . . . 10, 20, 30, 40, 50, 60, 70, 71, 72 [*successfully coordinates both kinds of tens and shifts to ones words and ones objects*].
Adult: Carolina says she got 72. What did you get, child tutor?
Child tutor: 72. Wait, wait (child tutor counts again).
Adult: It looks like you did it exactly the same. You took out two tens. How did you do it, Carolina?
Carolina: I saw that these are tens (dimes) and this is a ten (written 2 in 28 on paper) so I took away two (dimes) and these are not (pointing to written number 8 in 28 on paper). This is not a ten. So I took away eight of the pennies' cause this wasn't a ten. [*She describes her finding the marks number of tens and the tens objects and taking them away, and then finding the marks number of ones and the ones objects and taking them away.*]

## Example 13

6/24 Progress, but a new situation: Articulates conditions for a tens-to-ones shift in counting but does non-maximal (but correct) solution to counting answer as mixed dimes and pennies with the 7 ones pennies in the third of 6 stacks of pennies (all other stacks have ten pennies).

This whole long session continued the 6/23 work with dimes, ten stacks of pennies, and loose pennies. Several pieces of this session are included here to show the affective flavour of this tutoring, Carolina's increased confidence and ability to articulate aspects of the tens/ones shift, and a new situation requiring new competencies: either counting on by tens from a mixed sequence number (e.g. 27, 37, 47, 57) or counting by tens out of order (skipping over the pile of 7 pennies). [$1.00 − 43¢ is the problem at the end]

Carolina, child tutor B, adult

Adult: So here's some dimes. Here's some pennies. I want you to make your dollar not just out of dimes and not just out of pennies, but use some dimes and some pennies.
Carolina: Do we have to do that?
Adult: I know. That was a little hard for you, Carolina. That's why we're going to do it again today.
Child tutor: She was crying. But it's okay. If you get it wrong, it's okay. I got two of them wrong, and you got two of them right. And I got them two wrong. It doesn't matter [*provides emotional and ego support to Carolina*].
Adult: Yeah. And besides, if you need help, we'll just help. Okay? [*Provides encouragement.*]
Carolina: I think I'll get them right without a whole lot of helping [*displays increased confidence*].
Adult: Well, let's give it a try [*emphasises trying rather than errorless performance*].
Carolina: I'm going to make a dollar now [*signals end to need for emotional support, moves herself into the task*].



. . .
(Each girl has made her dollar with dimes and groups of ten pennies.)
Adult: Carolina, can you explain your dollar, please?
Carolina: I put six dimes and then I put ten pennies here and ten pennies here and ten pennies here and ten pennies here. And then I went 10, 20, 30, 40, 50, 60, 70, 80, 90, 100. Because if I only had three (pennies) that means that I would go 10, 20, 30, 40, 50, 60, 61, 62, 63—if I only had three. But if I put ten here, then I can go 10, 20, 30, 40, 50, 60, 70, 80, 90, 100 [*uses hypothetical quantity with some ones to explain why only counted by tens*].
Child tutor: See, that was pretty easy [*provides emotional support*].
Adult: Yeah, that was great. All right, are we all ready to go? (Adult clears the remainder of the money so it doesn't get mixed up.)
Child tutor: What are we going to do?
Adult: We're going to do problems now. We have a dollar, and each time we're going to have a dollar, so try not to mess your dollar up when you count. 'Cause we're going to use the problems where you have a dollar and you take away something and then the next problem we'll start all over again with a dollar. So, we'll have to put our dollar back together again after each problem. So make sure you don't get super-messy with your dollar [*specifies general problem types and a need for neatness to speed up work across problems*]. Oh, I see, child tutor is *stacking* pennies.
Carolina: Me, too.
(The girls stack their pennies.)
Carolina (explains why she stacks her pennies): If it's a take-away, then I can take one (stack) up and another up [*Carolina anticipates her second step in problem solving and describes how the stacks will help. So she is not just mindlessly copying the child tutor; she sees a purpose for stacking.*]
Adult: That's right. That's a good idea (adult helps Carolina make sure she has ten pennies in each stack) [*Carolina still sometimes makes counting errors*].

. . .
(The girls begin to subtract 43¢.)
Carolina: This (dime) is a ten, right? [*Carolina verifies quantity of a dime as ten.*]
Adult: Um/mm.

. . .
Carolina: Did I do mine right?
Adult: Yeah. So it's a dollar take away 43 cents.
Carolina: I did mine right. Don't be copying (to child tutor) [*Carolina demonstrates her success and our norm against copying*].
Adult: Okay, so, how much have you got left, Carolina?
Carolina: I have ten, twenty (2 stacks of ten pennies), twenty-one? [*Checking about her shift to ones; she is counting the stack of seven pennies so does need to shift to ones.*]
Adult: Show me again.
Carolina: Okay, wait, I have to put these (penny stacks) down. Ten, twenty (then counts all her pennies by ones, starting at twenty-one through 57. She messes up her ones counting one time and has to start over. During the second count she pauses before saying 40 and 50 but says the correct number).
Adult: Okay. Write your answer down right there (on the paper), and I'll get one (a piece of paper) for child tutor.
(Carolina writes 57 on the paper.)
Child tutor: Yeah. Mine was 57 too.
Adult: How did you do it, child tutor? Carolina, let's watch how child tutor did it, OK?
Child tutor (counts 1 to 7 by ones; counts 10 to 50 by tens): 57.

Adult: Oh. So you counted all your tens just as a ten instead of one, two, three [*sequence counting, not tens/ones counting, her more usual method*].
Child tutor; Um-mm. Instead of going like that.
Adult: Let's see if you can do it that way too, Carolina.
Child tutor: It's easy like that.
Adult: Let's see if you can count your tens by ten, okay, Carolina?
Carolina: 10, 20, 30 (point to the group of seven pennies) [*counts ones by tens*].
Adult: No, that's seven.
Carolina: 10, 20 [*again counts the ones by tens*].
Adult: No. You better count just these tens (pointing to penny stacks of ten).
Carolina: 10, 20, 30, 40, 50, 51–57 [*correct*].

. . .
(Both girls solve problem with sticks and dots.)
Adult (to child tutor): Okay, you got the same? Okay, both of you explain what you did on sticks and dots.
Carolina: I go first. I put 100 dots [*she actually drew 10 ten-sticks, but this shows her strong knowledge that these 10 sticks are actually 100 dots*].
Adult: Did you put 100 dots, Carolina?
Carolina: I mean I put 100 . . . (pause).
Adult: You put ten tens.
Carolina: Ten tens. Then I circled four, then I circled part of the three [*she circled part of one ten to show the 3 ones taken away*].
Adult: Part of the three or part of the ten?
Carolina: Part of the ten, and I went 3, 4, 5, 6, 7, 8, 9, 10 (making dots for 4 through 10 in the rest of that ten). Then I counted and went 10, 20, 30, 40, 50, 51–57, and that was the answer.

## Example 14

6/8 To estimate the answer, in subtraction, alternating focus on tens and ones quantities, and then coordinating these incorrectly

Both Carolina and child tutor A want to try to find how much change from $1 if an item costs 99¢. Adult asks child tutor to guess answer before calculating. [estimating $1 – 99¢]

Carolina, child tutor A, adult

(Child tutor guesses 1, then 10, then 11 [*focuses on the ones, then the tens, then coordinates these incorrectly*].)
(Carolina guesses 10, then 43 [*little notion of 99 as a quantity*].)
(Child tutor begins circling ones, realises answer is going to be 1, and wants to erase prior estimate.)

## Example 15

6/14 Child tutor B mentally constructs another ten in counting REST quantity

Carolina teaches; she chooses the problem 89¢ – 55¢. [89¢ – 55¢]

Carolina, child tutor B, adult

(Child tutor constructs 89 as 9 ten-strips with one penny covered. Takes away 5 ten-strips, asks for another yellow strip, covers five pennies on another ten-strip.)



Child tutor: 1, 2 [*counting REST ten-strips by ones as tens: 2 tens*].
Carolina: Go ten [*wants her to count by tens, not count the tens*].
Child tutor: 10, 20, 21 (is counting the 5 pennies left on a ten-strip), 22 (stops, changes her mind). 1–9 (ones on original ten-strip with one penny covered) 10–14 [*shifted from sequence counting to counting the ones separately*]. (moves to begin to write)
Carolina: Yes!
Child tutor: Whaaat?! [*Has to figure out 20 plus 14, the results of her sequence tens and separate ones counts because she did not count the ones on from the tens.*]
Child tutor (recounts 1–9, moves to 5 ones on other ten-strip): 10 (stops, thinks). So that's ten. That's one ten (counts rest of ones). 1, 2, 3, 4 (thinks).
Carolina: Nooooo! (this is not a correction, but distress that answer she expected may not be the right one).
Child tutor: OK. Then that's . . . 30. 31–34 (looks at adult, who nods) [*counts the extra mentally constructed ten on to her count of 20 to make 30 then counts the remaining ones on from 30*].
Child tutor: 'Cause I saw another ten [*reiterating that she made another ten out of the ones*].
Child tutor: . . . then I went 1–9, 10 pause, 11–14. But, I didn't realise how to do it, so I went again 1–10, I stopped, 1, 2, 3, 4. I thinked and goes 20, and another ten is 30, and then I went 30 (points to where she previously counted ten), 31–34. Got to do it in writing. [*child tutor conforming to the tutoring norm to explain how you get any answer*].