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GROUP CASE STUDIES OF SECOND GRADERS INVENTING MULTIDIGIT ADDITION PROCEDURES FOR BASE-TEN BLOCKS AND WRITTEN MARKS

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Small groups of second graders were asked to add horizontally presented 3- and 4-digit numbers using base-ten blocks and written marks recordings of the block procedures. Most children displayed increased conceptual understanding of place value and multidigit addition and demonstrated better written addition competence at the end of the five- to eight-day learning situation. The six groups displayed individual patterns of invention and learning that were dependent upon the personalities and mathematical understandings of the group members. Children easily added with the blocks, devising accurate strategies for multiunit sums of ten or more (e.g., twelve tens, sixteen ones, eleven hundreds). Many children did not spontaneously link the block addition to marks addition, instead operating in two separate worlds. When blocks addition was linked to marks addition, the blocks were a powerful support for conceptual understanding of marks addition. Blocks words were in some cases a more powerful support than were English words, and complete verbalization of trading seemed to be very helpful in facilitating understanding.

This article reports how, through complex intertwinings of personalities and children's different mathematical understandings in each of six groups, unique patterns of group interaction and paths of learning occurred. Six small groups of four or five second graders participated in this study, three during each of two data-gathering sessions. These children were in the second-grade math class that was the top of three in their school. They were assigned to groups that were balanced by gender and homogeneous with respect to conceptual and procedural competence in place value and multidigit addition and subtraction as assessed by pretests. An adult experimenter videotaped and took live notes of each group's meetings and guided their initial experience with base-ten blocks. During problem-solving, experimenters intervened to curtail rowdy behavior or to redirect sustained incorrect mathematical thinking. Children used digit cards during the first data-gathering session, a large paper pad ("magic pad") during the second session, and individual papers during both sessions to show their marks addition. See Fuson, Fraivillig, and Burghardt (1992) for further details of the methodology and early learning in the groups.

The focus of this report is on group case studies of the addition portion of the study. The analysis of children's mathematical interactions relies on the theory of multiunit understanding in Fuson (1990). Personality factors combined with the mathematical strength of individual children to create different group learning paths and different addition procedures with the blocks and the marks. Over half the children had had a first-grade teacher who used the blocks to teach place value but not addition, so the children were quite heterogeneous with respect to initial knowledge of the blocks. Children ranged on the pretest from solving no 2- to 4-digit addition problem correctly (6 children) to solving all problems correctly (4 children); they showed a similar range in place-value knowledge and conceptual explanations for 2-digit and 4-digit trading and alignment of uneven problems. On the posttest and/or in the videotaped data most of the children demonstrated increased conceptual knowledge concerning place value and multidigit addition and in the ability to do written multidigit addition (four children were at ceiling on the pretest written marks problems but showed increased understanding in some other task).

Official leader and checker roles rotated daily among children in a group. The intent of assigning these roles was to increase equality of participation among the children (Cohen, 1984). Children did respond to the "leader" roles by participating more actively in the groups' problem solving on their "leader" day, but "natural" leaders who led the group on most days also emerged in all groups.

Group 1: First session, high initial knowledge. These children, two girls and two boys, made few errors in adding numbers in written form on the pretest; they did their most interesting work with the blocks in subtraction (to be reported in another paper). During the study, they worked backwards from the written marks to the blocks and took two days to work out all of the details of relating their written procedure to addition with blocks, verbalizing their blocks addition, and showing the written marks procedure with the digit cards. On the first two days they made three vertically-aligned rows of blocks, one for each of the addends and one for the sum. They physically traded in ten of one kind of block for one of the next larger block, which they then put above the blocks of that kind (just as the 1 is written above the next left column in the standard U.S. procedure). The experimenter tried to get them to think of another way to add with the blocks or in written form, but they could not -- at this point they seemed to be too constrained by the standard written procedure. On the third day, however, they solved a problem by adding with the blocks from left to right and did the trading correctly. They set up and solved a 4-digit plus 3-digit problem correctly with the blocks but aligned the 3-digit number on the left with the digit cards and on individual papers. But because they solved the problem with the blocks and recorded the written solution from the blocks, their answer was correct. On their fourth day they were asked to use only the digit cards and to just talk about the blocks. They left-aligned a 4-digit plus 3-digit problem and got an answer that they recognized was too large. They figured out that the 3-digit number only had hundreds and therefore was aligned incorrectly. In response to urging from the experimenter, one girl invented a new digit card procedure in which she used the digit cards as named-value numerals (all numbers were made with extra zeroes to show their value: 2678 was made as 2000600708). This procedure was demonstrated and discussed on the final day of addition. The children agreed that this answer was too large (i.e., these are not standard written marks). The children worked together fairly well in this group with the exception of one boy who was quite disruptive and negative and repeatedly involved the other boy in physical and verbal disruption and picking on the girls. This was probably exacerbated by the fact that this addition work was too easy for these children; their behavior improved in subtraction especially with zeroes in the minuend.

Group 2: Second session, high initial knowledge. This group of two girls and two boys also made few pretest errors in written addition. From the beginning, these children vertically aligned the blocks. They disagreed about whether they should use separate blocks to show the sum or just push the addend blocks together. On the first day, they used extra blocks to show the sums of the ones and tens but not of the hundreds and thousands. They just counted the blocks in both addends on subsequent problems. This group began by adding the blocks from the right (as in the standard U.S. written procedure) and continued this for all problems. One child on the second day started adding blocks from the left, but he was stopped by the other members. For the first three days they did not physically trade the blocks when the sum exceeded nine, but recorded the trade in the written procedure and talked about how they could not write two digits and so had to trade ten to the next column. everyone agreed about the blocks and written procedures, but the explanations were not very full. The experimenter continued to say that they should do everything with the blocks that they did with the marks, but they did not seem to see the necessity of trading the blocks physically even though their explanations sometimes used block words and described block trades. On the fourth day the experimenter asked children to make explanations of their written marks procedures. The children spontaneously used block words and fully described the required block trades (e.g., saying "I took ten flatheads and put them together to make another fatty" to explain the 1 written above the thousands place). On the fifth day finally, on the last problem, the children spontaneously traded the actual blocks. They then exclaimed that they understood what the experimenter had meant when she asked them to do with the blocks what they had done with the written marks. That day, the group also progressed from aligning the first 3-digit and 4-digit problem on the left to solving another such problem without aligning digits at all, but adding the correct multiunits both with the blocks and digit cards. They aligned all subsequent uneven problems correctly. This group worked fairly well together though there was some antagonism between the mathematically strongest boy and girl. They were often distracted and silly, again perhaps because the problems were not very challenging to them. In general they continued to attend to their mathematical tasks at the same time as they carried on irrelevant discussions. They also went on to show their best work and thinking in subtraction.

Group 3: First session, medium initial knowledge. This group of two girls and two boys had two members O and M who worked hard at understanding addition with blocks and the digit cards, one boy D who had strong conceptual understanding but frequently dropped out of problem solving unless prompted by the experimenter, and one girl U who sometimes disrupted the group activities, except when she was the "leader." U gradually withdrew from group participation, with moments of engagement occurring late in the session (see burghardt, 1993, for a case study of this child). The group began by setting up the addends with blocks (second addend above the first) and adding the blocks mentally from the left to get the answer: three thousand twelve hundred sixty two (the ones column sum was twelve and was mentally added to the tens sum fifty). One child then showed the hundreds to thousands trade with the digit cards and described it in block words, saying you couldn't have twelve hundreds. Thus began five days during which the group quickly figured out how to add the blocks, trading correctly moving either from the left or from the right, but floundered with the digit cards, inventing several wrong marks procedures as well as frequently using the correct standard procedure of writing a 1 above the next left column. (See Fuson & Burghardt, 1993, for a report of these incorrect procedures.) During this time they did not link the blocks addition closely to the digit card procedures, and they discussed the digit card procedures only in terms of digits or English words, rarely in block words. The experimenter on the sixth addition day forced the children to link the blocks and

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the digit cards at each addition step and to describe digit card moves using block words, and the group agreed on a correct procedure. On the next day O again showed confusion when adding the tens but corrected himself when the experimenter asked him to think about the blocks. O then suggested a new marks procedure in which the top addend is increased by one (e.g., a digit card 5 is replaced by a 6) if a block needs to be traded to that column because that is how the children did it with the blocks: they put the new block in with the top addend blocks (this is actually the written procedure that was learned by second graders in Fuson, 1986). Discussion continued that whole day comparing O's new written solution and the standard method of writing the traded 1 above the addend. O was confused on one more problem about the trading of too many tens and convinced D using the English words "tens and ones," but M used block words (with the experimenter's support to withstand the boys) to establish the correct trade to the hundreds. The mathematical work would have progressed more smoothly if D had been more dominant and if U had not been so disruptive; U began with good written competence but learned little during this study because she was physically or attentionally absent from so much of the group activity.

Group 4: Second session, medium initial knowledge. This group consisted of three girls with one or no errors on the pretest written addition problems, and two boys who did no written problem correct on the pretest. Overall the group was enthusiastic and worked well together. They spent much of the second day and some of the third struggling to write a block number they had made with twenty nine teeth (unit cubes). They recognized that writing two digits for the units would make the wrong number and suggested many different nonstandard notations to show this number (e.g., 3 4 29 or 3 | 4 | 29). The experimenter finally asked them if they could make any exchanges with the teeth and the licorice. This group then invented a written procedure in which they added each kind of multiunit (ones, tens, hundreds, thousands), wrote the sum in two digits if necessary, and then fixed this answer to be in standard notation using only one digit per multiunit. This procedure evolved from their use of blocks: they first added each kind of multiunit, recorded their sum with marks (e.g., 3 12 5 12), traded ten of any blocks that had ten or more for one of the next larger block, and recorded the successive fixed sums (e.g., 5 12 5 15 became 6 2 5 15 and then 6 2 6 5). Some children continued to write problems in horizontal form throughout, while others wrote problems vertically aligned. At this point, all children understood addition and their written marks fixing procedure conceptually when supported by blocks, but none were able to "fix" sums without the support of blocks. On the following day, however, the group talked themselves through the trades using marks only and successfully fixed an answer. Some children continued to work on devising and understanding a fixing method for their written marks procedure during the final two days, describing what they were doing with block words and using the blocks when necessary. The fixing usually proceeded from left to right. Others devised a general method of fixing that did not depend on talking through the fixing with block words: for the 2-digit sums they crossed out the 1 and wrote a 1 above the next left digit. When the group had to move on to subtraction, all but one child could carry out their invented add-first-fix the sum method with written marks only and could explain this procedure in terms of trading multiunits. This group worked well together partly because the two most dominant members (one boy and one girl) were exemplary "good" rather than "bossy" leaders and had the strongest mathematical knowledge.

Group 5: First session, low initial knowledge. Of this group of two girls and two boys, one girl, M, was the dominant group member. M had little initial conceptual and procedural knowledge, while the others showed moderate to perfect pretest performance on written addition solutions. Through the first three days of addition the children, led by D and X, worked toward a blocks and digit-card procedure in which the blocks were aligned vertically and sums over nine had ten of that block traded for one block in the next left column. There was disagreement about the order of making the addends and whether to add from the left or from the right (each was done on different problems). Descriptions and explanations sometimes centered on the number of blocks and omitted the kind of block, leading to errors and prolonged discussion, and full verbalizations of the block trading were not given (they focused either on the new one ten or hundred or the old ten ones or ten tens but did not verbally describe the ten tinies traded for the one rectangle). Over the next four days M invented and imposed a new procedure in which the goal was to leave only nine in a given column (because "you can't have more than 9 in a column"); the excess over nine (or sometimes over ten) was taken away. This excess was often put above the next left column, but was sometimes dropped (M's procedure led to answers like 6999 or 4999). This 9's procedure competed with the ten-for-one trading procedure over four days, with children frequently using the 9's procedure with the blocks and the digit cards, and the standard algorithm on their individual worksheets. All four children changed their views repeatedly within and over days, frequently expressing confusion. During this confused period, children talked about how many they had to take away from column sums over nine to make that sum small enough. On the third such day the experimenter encouraged the children to keep the blocks and the written marks connected and reviewed the ten-for-one trades with blocks. Over the final two days of addition, the experimenter continued to support linking the block and written marks procedures and queried the children about the size of the blocks. The children eliminated their 9's procedure in favor of their written trading procedure. At the end, all of these children were able to verbalize some understanding of the correct ten-for-one trading, although their explanations were still incomplete.

Group 6: Second session, low initial knowledge. This group of three girls and two boys ranged on the pretest written addition tasks from making only one fact error to getting all the sums wrong. During the first few days, the boys and girls argued about how to write and solve problems, but soon the girls became established as the most actively engaged members of the group, and the boys deferred to the girls. The group presented the first problem horizontally with the blocks and then added the blocks beginning with the thousands. When they got a sum of twelve breads (hundreds), one child said that there couldn't be two numbers in the sum so "you put the two down and add the one to the top of the other side." This verbal description arose from procedural knowledge of the standard written algorithm, but did not specify sufficiently where the "one" should be written. Because they were moving from left to right some children wanted to write the 1 above the next column, i.e., at the top of the tens column. Confusion over where to write the 1 persisted over the next four problems. "Regrouping" was referred to as a written method unrelated to the blocks; it had to do with writing the 1 somewhere. Children wrote the next problems vertically, some asserting that you can't regroup with horizontal problems. For the next two days, this group continued to add blocks from the left and write the 1 above the column to the right, in the mirror image of the trades that they had been previously taught. Soon, however, the girls said that the 1's were wrong because they were adding in the wrong direction ("You move to the left, the opposite of writing"), but the boys continued to insist on adding from the left. One child focussed on the size of the blocks representing each trade, so the blocks were traded to the next larger column. On the fourth day of addition, addition began from the right and, on the fifth day, the experimenter asked the children to add one problem both from the left and the right. This produced two different solutions, a correctly traded answer from the right and an incorrect solution from the left due to a mirror-image trade. Although the children had previously traded correctly when adding from the left, the mirror-image trade occurred when a child allowed her written addition to dictate the blocks trade. A heated discussion followed and, from then on, the children added from either the left or the right flexibly, using blocks, trading correctly, and recording correctly. The girls by the end all had given conceptual explanations for various blocks trades but, when helping the boys, more often gave procedural explanations to them. On the eighth day, the experimenter asked the children to do a problem on the magic pad and explain it by talking about the blocks. The girls could all do so, but the boys required help. One of the boys was very shy throughout, and the other boy frequently withdrew from active participation.

Discussion

A striking aspect of all of the group work was the relative ease with which children invented accurate quantitatively-based multiunit addition with the blocks compared to the many inaccurate invented multidigit written marks procedures (see Fuson and Burghardt, 1993). Children never added different block multiunits but did add written digits for different multiunits (e.g., hundreds and thousands). The block quantities also suggested what to do when children had too many in the sum of a given multiunit (e.g., twelve tens) and provided language to convey the quantities involved in these solutions. The written digits instead elicited nonquantitative procedural language ("Write the 1 up there") even when the digits were being used to describe block moves. Using block words (e.g., tiny, long legs) to describe written digit procedures was sometimes more helpful than using English words (one, ten) because the block words require a child to be clear about both the kind of multiunit and how many multiunits there are. These can get confused in English: a child would say "ten" to mean either "ten ones" or "one ten," but had to say "a tiny" or "ten tinies" or "one long legs" when using block words. The ambiguities in English led to confused communication among children and allowed erroneous written procedures; blocks and block words clarified these confusions. Many children did not spontaneously link blocks addition and written marks addition, resulting in erroneous written marks procedures. When experimenters forced children to link the blocks and written marks for each multiunit (e.g., children had to write the hundreds marks as soon as they added breads), the quantities in the blocks enabled children to correct their written marks procedures. Verbalizing what had been done with the blocks, especially with blocks words, also proved to be helpful to some groups in correcting written procedures.

Personalities and mathematical knowledge both contributed to the quality of the mathematical work of a group. When dominant members had good mathematical knowledge and were good rather than bossy leaders, the groups made better mathematical progress. Most groups were not very good at identifying group members with inadequate understanding, and some such members hid their lack of knowledge fairly successfully. More focus on such helping, a longer time on addition for some groups, and more time to do backwards linking with everyone discussing the marks procedures in blocks words would have helped the weakest children. The strongest children could have handled more difficult questions such as "What are differences between adding from the right and from the left?" Second graders can do interesting mathematical work in this environment, but they do need some help from a teacher to maximize their use of group work, to relate the block quantities to written digit procedures, and to verbalize their solutions conceptually. We are presently analyzing data from low- and middle-achieving children to see how these results generalize.

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