In D. L. Haury (Ed., 2002). Proceedings of the twenty-fourth annual meeting of North American chapter of the International Group of the Psychology of Mathematics Education (pp. 1641 – 1649). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

# A LENGTH MODEL OF FRACTIONS PUTS MULTIPLICATION OF FRACTIONS IN THE LEARNING ZONE OF FIFTH GRADERS

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We developed and implemented a curricular unit concerning operations on fractions. The unit used drawings of lengths fractured into equal parts for all operations. Students initially had 2 interpretations for multiplying by a unit fraction (i.e.,$1/n$), one that generalized and one that did not. The predominant method, dividing by equal shares, did not generalize when the denominator of the unit fraction did not divide into the whole number evenly. The second interpretation, which did generalize to a unit fraction or a non-unit fraction times any whole number, was taking the unit fraction of each "1" (or single unit) in the whole number and then finding the sum of these products. The class and teacher then extended this second method to multiplying a fraction times a fraction, relating via written and oral explanations the length drawings to a general written algorithm of multiplying the top numbers and multiplying the bottom numbers. Students outperformed samples of U.S. students using traditional textbooks and Japanese and Chinese fifth graders.

## Purposes, Perspectives, and Theoretical Framework

Multiplication of fractions is on the Grade 5 curriculum in many states, but tests indicate that most U.S. students using traditional textbooks cannot solve such problems (Stigler, Lee, & Stephenson, 1990). Most East Asian students also do not solve such problems accurately (Stigler, Lee, & Stephenson, 1990). The question then arises whether this topic is developmentally inappropriate or whether the usual approaches to such teaching are not effective and fifth graders could learn with an effective approach.

A common visual representation of fractions used in instruction is a circular (e.g., pie, pizza) model of fractions (for discussion see, for example, Behr, Harel, Post, & Lesh, 1992; Clements & Del Campo, 1988; Lamon, 1999; Moss & Case, 1999). Pies and pizzas are within the experiences of most students, but both the real-world objects and the circular drawings of fractions frequently violate the central characteristic of fractions: fractions must have equal parts (equal shares). It is not easy to divide a circular model into equal parts except for 2, 4, and perhaps 8 parts. Examples of errors in circular drawings made by students at the beginning of the present study are given in Figure 1.

We instead propose a length model as a generalizable model that is easier to partition into an arbitrary number of equal parts and that will support all of the concepts and operations on fractions. Length models are also good for decimals and

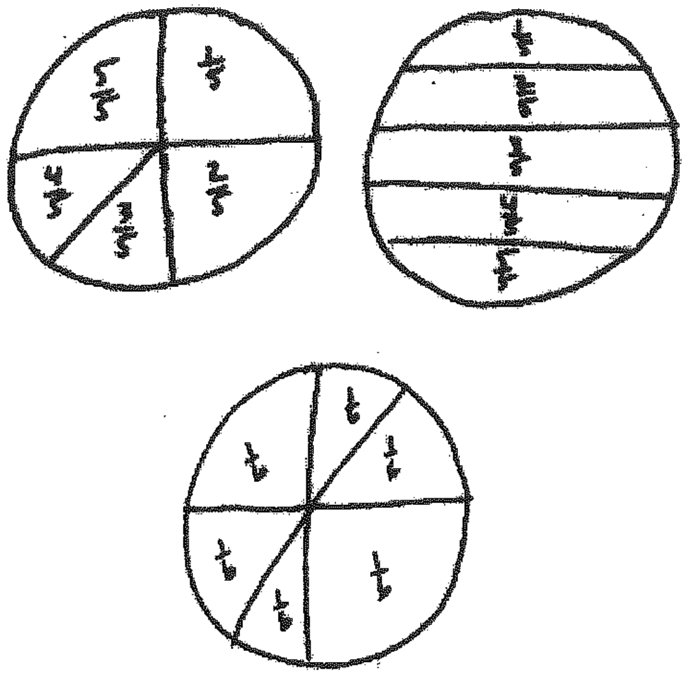

*Figure 1.* Unequal fractions in student circular models.

for metric measurement, so they would also enable fractions to be related to decimals and metric lengths. In this paper we focus on multiplication of fractions, and we examine whether a length model will pull multiplication of fractions into the learning zone of fifth graders.

Our theoretical framework uses both a Piagetian constructivist model of learning and a Vygotiskiian socio-cultural model of teaching. From our Piagetian perspective, we assume that students are continually interpreting their classroom experiences using their own conceptual structures as well as continually adapting their conceptual structures to their on-going classroom experiences. From our Vygotskiian perspective, we assume that a major goal of school mathematics teaching is to assist learners in coming to understand and use cultural mathematics tools. One means of assistance is drawings, which are semiotic tools that can support sense-making both individually and in the classroom discourse about mathematical thinking.

Our research question is whether our length models fit the learning zones of fifth graders well enough that length model drawings can be used to develop general methods of multiplying with drawings, which can be related to general numerical methods for multiplying fractions. This study is part of a larger project 1) examining the utility of the length model for helping students build understanding of all the concepts, situations, and operations involving fractions, 2) seeking to identify students' general learning paths when using a length model, and 3) understanding how this approach to fraction understanding relates to concepts in other multiplicative domains (e.g., functions and ratio and proportion).

## Methods and Data Sources

Participants were 25 fifth-grade students in a classroom in a mid-western multi-racial small city with a considerable number of immigrants, a large minority of the students on free lunch, and a substantial number of students with highly-educated parents. This diversity was chosen to test the accessibility of our approach to a broad range of students.

The teacher, Ms. H., was an experienced teacher of reform mathematics who was recommended by a district-wise administrator for the quality of the mathematical discourse in her classroom. Working with such a teacher ensured that instruction would involve sense-making by all students and that alternative student methods using the length model drawings linked to numerical methods would be discussed. The students used the *Everyday Mathematics* curriculum as their regular mathematics program and were used to discussing their methods.

The present fraction unit was developed by the authors as an alternative approach to the one suggested in the *Everyday Mathematics* curriculum. In total, the fractions unit involved approximately an hour a day of class-time for a total of 15 days. Four days were spent on the multiplication of fractions. These days are described below in the results and conclusion.

The authors co-developed the initial version of the multiplication of fractions approach. The second author and Ms. H. met regularly to discuss the ideas in the approach, to address any concerns of either of the authors or of Ms. H., to make any necessary changes to adapt the unit to district goals and classroom idiosyncrasies, and to assess and adapt to on-going student progress. The second author, who is also an experienced elementary teacher, went to each class, videotaped each class, and contributed to the instruction and discussion when appropriate. The students understood her to be a co-teacher of the unit and someone with whom they could talk, of whom they could ask questions, and from whom they could seek help.

Data for analysis of the classroom instruction were videotapes, notes taken during and after class, and copies of the overheads made by Ms. H. and students during class discussions. Data for student learning were interviews of target students, students' work, and a test given at the end of the unit. This test was comprised of numerical items comparable to and word problems identical to those given by Stigler, Lee, and Stevenson (1990) to U.S. fifth graders using traditional textbooks and to Japanese and Taiwanese fifth graders. The test included items on most fraction concepts and was given 5 days after the multiplication teaching was ended and following one day in which all of the concepts were reviewed.

**Results**

On the first day, students' methods were elicited for simple multiplication problems in which a unit fraction would divide a whole number evenly (e.g., 1/3 x 6). Students had two different interpretations of such fraction times a whole number problems (see Figure 2). Some students used an equal sharing notion of multiplying by a unit fraction. They found "one-third of (the whole group of) 6" and so divided 6 into 3 equal groups of 2. Others found "six one-thirds" or "six one-thirds of one" and then accumulated these thirds numerically or on a length drawing to get six thirds, which totaled 2 (thus 6 was formed by two groups of 3 thirds).

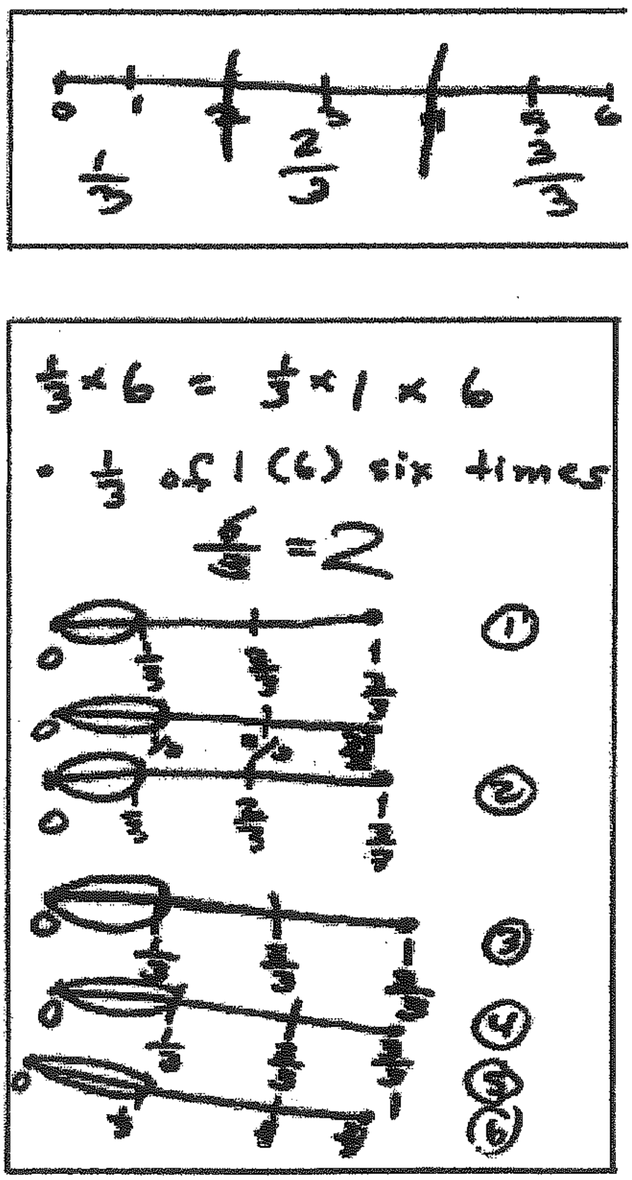


*Figure 2.* Student Length Models showing 2 interpretations of the Meaning of 1/3 x 6.

Instruction then moved to more difficult problems in which the fraction did not divide the whole number evenly. Students were unable to extend their own methods to these more difficult kinds of problems because most of their initial methods depended on the idea of a fraction as the equal sharing of a whole number. Students could not proceed if the fraction did not divide the whole number evenly. Using the length model, the piece-by-piece multiplying method that had arisen on Day 1 for 1/3 x 6 (see bottom of Figure 2) was extended first to a fraction times a whole number (Day 2) and then to a general fraction times a fraction case on Day 3. Each of these extensions was made in a whole-class co-constructing session led by the teacher with considerable input from students. Students then chose their own "fraction times a fraction" problem, drew a length model for their problem (see an example in Figure 3), and wrote an explanation of their step-by-step thinking.

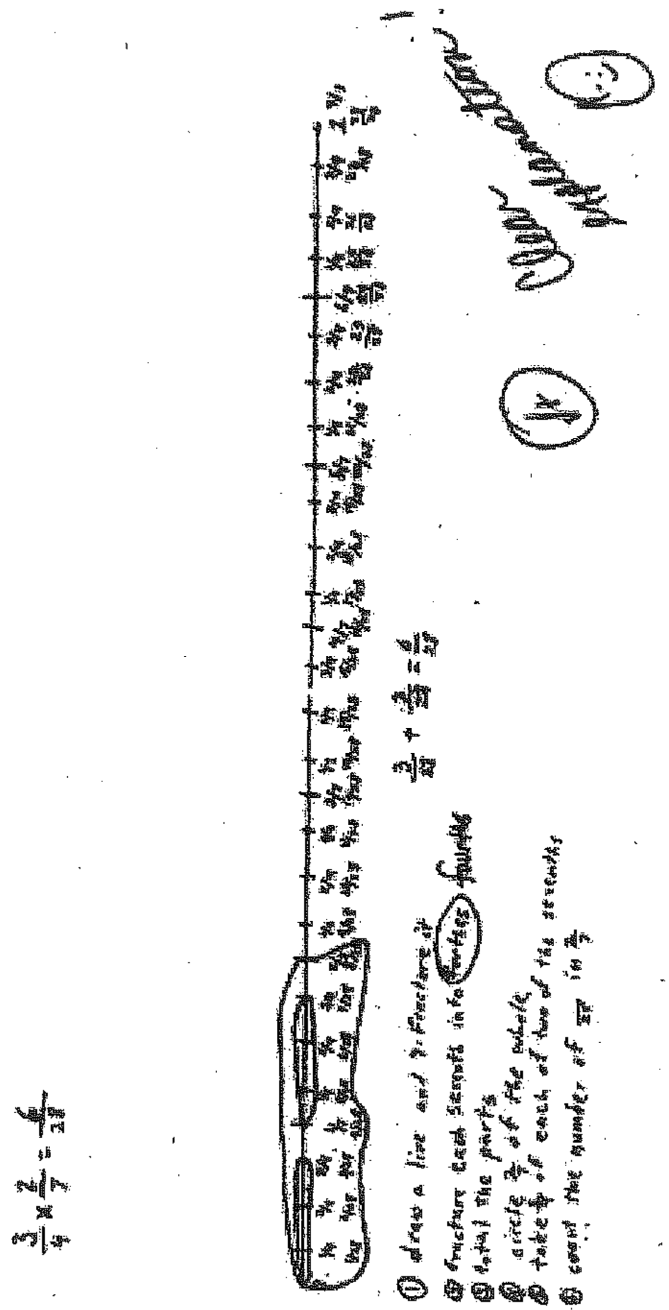

*Figure 3.* A student drawing and explanation of 3/4 x 2/7.

An example of a student explanation at the overhead projector of multiplying a fraction times a fractions is as follows:

JT: Here's my fraction, four-fifth times two thirds (writes 4/5 x 2/3 on a transparency). So I'm going to draw a line five segments long...(draws a horizontal line five inches long and places hash marks at each inch along the line). So now that I've labeled my fifths I'm gonna divided them each into thirds (places three evenly-spaced hash marks between the five-fracturing hash marks he drew previously). So I divided each fifth into, I divided it into thirds. [So now JT has done 1/5 x 1/3, the bottom part of the algorithm.] Now I have to circle four-fifths. That's the number of parts I have. Four-fifths is my fraction, so I circled that (retracing the large circle he made around four-fifths of the original line). Um...so now I have, I have to circle every two thirds (pause) because I have two-thirds times four-fifths. I circled the four-fifths, now I have to circle the two-thirds (pause). And you don't circle past the circled part (referring to the large circle around four-fifths). That's your extra. You don't need that. If you count every third, one, two, three (pointing to hash marks on his line) every, um, three thirds.

Ms. H: And you have five sets of three thirds? [seeking to get explicit naming and labelling of the common fracture: 1/5 x 1/3 = 1/15]

JT: Ya.

Ms. H: Which is a total of how many spaces?

JT: Fifteen.

Ms. H: Fifteen for your whole, right? For your whole unit.

JT: So, I totaled my whole and that was fifteen (writing "15" below his line). So now I already circled it [the two-thirds] so I have one, two, three, four, with two in each (pointing to circled sets of 2/3). Four times two equals eight (writing the equation vertically) and you have to multiply that because you labeled it two-thirds. You circled two-thirds for each one. You have four sets of two-thirds. [explaining the meaning for the top part of the multiplication algorithm] With two in each and that gives you a total of eight (writes 8 above the 15 he wrote earlier).

The fourth day focused on doing more examples and discussing why the algorithm of multiplying top numbers and multiplying bottom numbers worked. These explanations were related to the fracturing experiences students had when making the drawings, for example, "For the bottom, you fractured one fraction by the other fraction, so you get a fraction that is their product; for the top, you are taking the top number of groups of the size of the other top number, so again you multiply."

Over the 4 days in which students drew length models of fraction multiplication problems, almost all students were able to make such drawings correctly. Some required help along the way with various steps, and not all students could explain all of the steps as clearly as did the student in Figure 3 or at the overhead. However, all students were using the length models with most of the steps drawn correctly and linked to accurate numerical labeling.

The students did very well on the multiplication items given at the end of the unit (see Table 1). More than 80% of our length-model students solved the numerical fraction computation problems correctly, compared to means of 20%, 21%, and 14% in Japan, Taiwan, and the U.S., respectively. Our length-model students also did comparatively well on the word problems, with 63% correct answers compared to 54%, 49%, and 20%, respectively.

Comments by students during whole-class discussion and individually indicated that particularly powerful parts of the length model seemed to be its generality (students could choose their own fraction times a fraction problem and draw it with the length model), its affordance of seeing and writing multiplication as repeated addition (see the 3/28 + 3/28 in Figure 3), and the visual ease of the connections of the partitionings in the length model to the multiplications in the numerical algorithm.

**Table 1.** Percentage Correct on Multiplication Problems by Students in Japan, Taiwain, and the United States

| | Background of Students | | | |
|---|---|---|---|---|
| Item | Japan | Taiwan | U.S. Traditional | U.S. Length Model |
| Whole Number X Fraction<br>Fraction X Whole Number | 14 | 8 | 2 | 85 |
| Fraction X Fraction | 25 | 33 | 26 | 88 |
| Dad cut a cake into 16 pieces. George ate one fourth of them. How many pieces were left? | 65 | 63 | 30 | 56 |
| A stamp collecting club has 24 members. Five-sixths of the members collect only foreign stamps. How many members collect only foreign stamps? | 43 | 35 | 9 | 69 |

## Conclusions

Multiplying fractions by fractions is within the learning zone of fifth graders if a length model and a piece-by-piece partitioning method is used in a sense-making classroom environment in which drawings are referents for discussions and explanations of student thinking and the multiplying tops and bottoms algorithm is based on experience with these drawings. The piece-by-piece multiplying method works because one can gather the unit fractions into any mixed number (e.g., 1/3 x 7 instead of 1/3 x 6 is just one more 1/3, i.e., 2 1/3). The length drawings for a unit fraction times a whole number (e.g., 1/3 x 6) are the same as those for a unit fraction times a unit fraction (e.g., 1/3 x 1/6) except that the labelling is different (labelling 1 through 6 instead of 1/6 through 6/6). This similarity simplifies the transition from multiplication by a fraction x whole numbers to fraction x fractions.

A recurring issue throughout the unit is why does "of" mean "times"? Students had no trouble understanding "one-third of 6" as dividing 6 into 3 equal parts, but this experience then made them think that they were dividing, not multiplying. In fact, multiplying by a unit fraction is dividing by that whole number, but the operation *within fractions* is multiplication. Either multiplication grouping or comparing language can provide a basis for understanding why "of" means times for fractions as well as for whole numbers. Across different countries, people interpret 4 x 2 in two ways: as "4 sets of 2" or as "4 taken 2 times." Using both of these meanings relates "of" and "times" within the English language. So 4/5 x 2/3 can mean "4/5 of 2/3" or "4/5 taken 2/3 times." Within multiplicative comparing situations, English shifts from the "times as many" to " fractional parts of" language, again providing a relationship between these two. For example, we say, "Joe has 3 times as many as Mary has" but "Mary has 1/3 of Joe's amount."

Finally, we wish to highlight our view that 4 days is not sufficient for mastery of fraction multiplication. These days were part of a larger coherent approach to fractions using length drawings. Follow-up work would also be necessary to maintain the understanding built during this unit. We see three phases in building understandings in such a complex domain. First, the domain is approached using intuitive easy numbers in situations where objects or drawings can help students develop meanings. Second, these meanings are connected to generalizable numerical methods through discussion and linking to drawings. Third, a longer period follows of remembering and explaining in which occasional practice with numeric methods by students is accompanied by explaining why the method works. This phase is required to keep the meanings connected to the general numeric meanings. The first phase must be done with a view to the second and third phases. We have seen in this study how the use of easy intuitive numbers that divided easily led students to methods of multiplying by a unit fraction that did not generalize to numbers that were not evenly divisible. Curriculum development must keep all of these phases in mind from the beginning if students are not to be led to develop methods that will not generalize.

**Note**

The Children's Math Worlds research was supported by the National Science Foundation under Grant No. REC-9806020. The opinions expressed in this chapter are those of the authors and do not necessarily reflect the views of NSF.

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