

Teaching and Learning Two-Digit Multiplication: Coordinating Analyses of Classroom Practices and Individual Student Learning

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This article coordinates an analysis of 2-digit multiplication instruction in 1 U.S. 4th-grade classroom with an analysis of learning accomplished by a cross-section of students from the same classroom. In particular, the article compares how taken-as-shared classroom mathematical practices and individual students used features of rectangular area representations for accomplishing similar problem-solving goals. The analysis demonstrates (a) how classroom practices converged on methods that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property and (b) that some students accomplished similar coordinations by the end of the unit, whereas others still struggled to connect representational features with the goal of determining areas. The article provides a model for further studies that coordinate analyses of classroom interactions with analyses of individual student learning and suggests the detail with such analyses need to be conducted, if they are to provide insight into processes of teaching and learning with multiple representations.

This article uses connections between two-digit multiplication and rectangular area as a context in which to address two broad questions central to research in mathematics education. First, how can students develop conceptual and procedural understandings of core topics? Second, what relationships exist between classroom interactions and individual student learning?

Multidigit multiplication is an important, though understudied, area of research for two reasons. First, conceptual understanding rests on two mathematical coordinations required for extending multiplication from single- to multidigit

numbers. The first coordination is between magnitudes of factors and magnitudes of products. Students must learn when magnitudes of factors and products are the same (e.g., $2 \times 3 = 6$), and when they are different (e.g., $2 \times 30 = 60$ and $20 \times 30 = 600$). The second coordination is between expanded forms for factors (e.g., $20 + 8$) and the distributive property. Efficient multiplication methods that generalize to arbitrary numbers of digits rely on multiplying each term in the expanded form for one factor by all terms in the expanded form for the second, whether or not expanded forms are made explicit. If students develop initial understandings of the distributive property in the context of whole-number multiplication, they will be better prepared to apply the property in other domains such as fractions and algebra. The instructional materials at the center of this study used areas of rectangles and an expanded numeric method to support these two coordinations.

The second reason that research on multidigit multiplication is important is that U.S. students have performed poorly on multidigit multiplication items in international studies. Stigler, Lee, and Stevenson (1990) reported that only 54% of U.S. fifth-grade students in "traditional" courses could solve 45×26 correctly, and Mullis et al. (1997) reported that only 46% of U.S. fourth-grade students could correctly answer a multiple choice item that asked how much more 25×18 is than 24×18 (percentage correct for fourth-grade students from other countries included Hong Kong, 63%, Korea, 80%, and Singapore, 73%). The classroom analyzed in this article is of interest not only because of the instructional materials, but also because the U.S. fourth-grade students in this study outperformed the U.S. fifth-grade students in the Stigler et al. (1990) study: 80% solved 45×26 correctly on the end-of-unit test.

In the next section, this article reviews results from research on multiplication, the use of external representations in problem solving, and classroom microcultures on which this study builds. The article then describes the multiplication materials and uses them to illustrate the theoretical perspective on representations and problem solving that is used to coordinate an analysis of whole-class solutions with an analysis of individual student strategies. To understand students' opportunities to learn with the instructional materials, the article analyzes the evolution of taken-as-shared problem-solving strategies that arose through whole-class discussions over the course of the entire two-digit multiplication unit. Then, to understand those aspects of taken-as-shared strategies that students incorporated into their own problem solving, the article analyzes strategies that a cross-section of students used to solve similar problems during end-of-unit interviews. Both analyses focus on ways that representational features were used to accomplish problem-solving goals. Taken together, the analyses lead to results about opportunities and challenges when teaching and learning conceptual and procedural aspects of multidigit multiplication with area representations, and provides a model for future studies that coordinate analyses of classroom interactions with analyses of individual student learning. Such co-

ordinated analyses at the level of strategies are still rare (see Lobato, Ellis, & Muñoz, 2003, for a related example).

BACKGROUND

Classifications of situations that can be modeled by multiplication have consistently included rectangular areas (e.g., Greer, 1992; Schmidt & Weiser, 1995; Schwartz, 1988; Vergnaud, 1983, 1988). Research on multiplication has used rectangles to illustrate multiplication of fractions and the commutative property (Greer, 1992) and to investigate preservice teachers' quantitative reasoning (Simon & Blume, 1994). To the best of my knowledge, however, research has not examined the use of rectangular areas to develop students' understanding of multidigit multiplication. Moreover, just a few studies have examined students' understandings of the distributive property. In addition to Lampert's (1986a, 1986b) work discussed in the following, Matz (1982) reported high-school and college students' difficulties with the distributive property when manipulating algebraic expressions, and Weaver (1973) reported fourth- through seventh-grade students' difficulties on tasks such as $(3 \times 8) + (9 \times 8) = _ \times _$ and "3 sets of 8 and 9 sets of 8 are $_$ sets of $_$."

Classroom Studies of Multiplication

Research on multiplication contains several classroom-based studies (Confrey & Scarano, 1995; Lampert, 1986a, 1986b; Mechmandarov, 1987, as discussed by Neshet, 1988; Scarano & Confrey, 1996; Treffers, 1987). Of these, only Lampert (1986a, 1986b) and Treffers (1987) focused on multidigit multiplication. Lampert (1986a, 1986b) gave a detailed description of her instruction in one fourth-grade classroom that used coins, drawings of situations described in word problems, and numeric methods that separated partial products from multidigit addition. Although Lampert used drawings of situations to help students develop intuitive and concrete understandings of the distributive property, she did not include rectangular area situations and did not investigate how individual students understood and used the representations that she discussed with her class. Treffers (1987) described an instructional approach for third grade based on progressive schematization in which students solved multidigit multiplication problems using numeric methods that began with and then abbreviated repeated addition. Treffers did not describe possible connections between numeric methods and drawn representations and did not analyze classroom interactions between teacher and student. In contrast to the work of Lampert (1986a, 1986b) and of Treffers (1987), this study analyzes not only classroom instruction but also understanding that individual students used to solve multidigit multiplication at the end of that instruction.

Rectangular Areas and Arrays

Although some research has suggested that students connect multiplication to areas of rectangles by reciting, but not understanding, the length times width formula (De Corte, Verschaffel, & Van Coillie, 1988; Nesher, 1992; Peled & Nesher, 1988; Simon & Blume, 1994), these difficulties may be most acute when students try to understand relationships between lengths and areas as *measurements*. Simon and Blume argued that most learners must use rectangular areas, understood as arrays of unit squares, as the basis for understanding the transformation through multiplication of length into area measurements. Other research has suggested that understanding rectangular areas as arrays of unit squares can be accessible to upper elementary students. Peled and Nesher found that fifth- and sixth-grade students had good understandings of the constraints that equal-groups problems must satisfy, including situations where discrete objects are arranged in arrays. Students knew that rows and columns in arrays must have the same number of elements, but they could not connect rectangular areas to arrays or repeated addition when unit squares were not rendered. Reynolds and Wheatley (1996) found evidence that fourth-grade students understood rectangular coverings in terms of rows of unit squares that form composite units. In subsequent research, Outhred and Mitchelmore (2000) examined how first- through fourth-grade students covered rectangles by drawing unit squares of a size specified in each of three tasks. Although none of the students had been taught area measurement, virtually all fourth-grade students generated correct coverings with equal numbers of unit squares in each row and column. Battista, Clements, Arnoff, Battista, and Borrow (1998) suggested, however, that array structures are less accessible to early elementary students. These results suggest that fourth-grade students, like those in this study, may be able to use understandings of arrays to develop understandings of multidigit multiplication.

Counting and Repeated Addition as a Basis for Whole-Number Multiplication

Determining areas of rectangles understood as numbers of unit squares may be accessible to students because the array structure can support a range of strategies. Some researchers (Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1997) have analyzed how elementary-school students, typically in first- through third-grade, used blocks and other manipulatives to solve single-digit multiplication problems about equal groups situations using increasingly efficient counting strategies that led to repeated addition and culminated in recalled multiplication facts. Although whole-number multiplication is often introduced as repeated addition, there has been debate among researchers about the psychological relationship between the two operations. On one hand, Fischbein, Deri, Nello, and Marino (1985) argued

that primitive models for each arithmetic operation mediate students' selection of operations when solving problems, and that repeated addition is the primitive model for multiplication. Supporting evidence came from Bell, Fischbein, and Greer (1984) who found that 12- and 13-year olds performed better on word problem tasks when the multiplier was a whole number, so that multiplication could be thought of as repeated addition. On the other hand, Bell, Greer, Grimison, and Mangan (1989), Neshet (1988, 1992), and Peled and Neshet (1988) questioned Fischbein et al.'s position, arguing that students' experiences with language and school affect connections they make between multiplication and addition. Moreover, several researchers (e.g., Clark & Kamii, 1996; Confrey, 1994; Confrey & Smith, 1994, 1995; Schwartz, 1988; Steffe, 1988, 1994; Vergnaud, 1983, 1988) argued that different psychological operations and types of quantities are involved in multiplicative and additive thinking, and Steffe (1988, 1994) has traced the development of psychological operations for multiplication out of children's counting schemes. The range of strategies that arrays afford for determining rectangular areas, some based on additive thinking and some on multiplicative thinking, meant that even if students in this study had not developed multiplicative reasoning fully, they could still develop numeric methods that coordinate (a) magnitudes of factors and products and (b) expanded forms for factors and the distributive property.

External Representations and Problem Solving

Theoretical research on external representations has identified psychological processes for constructing and interpreting representations—including encoding, reading, syntactic elaboration, and semantic elaboration (Kaput, 1991)—and further psychological processes for using multiple representations—including translation (Janvier, 1987; Lesh, Post, & Behr, 1987). Some empirical research, particularly in the functions and algebra literature, has focused on students' evolving use of representational features for accomplishing problem-solving goals (e.g., Izsák, 2000, in press; Lobato et al., 2003; Lobato & Siebert, 2002; Monk & Nemirovsky, 1994; Moschkovich, 1998; Nemirovsky, 1994; Schoenfeld, Smith, & Arcavi, 1993). These studies have documented cases in which understanding how to use y -intercepts, slopes of linear graphs and equations, and other representational features for solving problems has been a significant accomplishment for students. Such research has examined connections between representations and problem solving at a finer grain-size than studies within the multiplication literature and motivates the detail with which this study analyzes whole-class solutions and individual student strategies.

Classroom Microcultures and Representational Practices

Recent classroom research has examined how taken-as-shared interpretations and uses of inscriptions can be established in mathematics classrooms. Cobb and col-

leagues (e.g., Bowers, Cobb, & McClain, 1999; Cobb, 1999; Cobb, Stephan, McClain, & Gravemeijer, 2001; McClain & Cobb, 2001) developed an *emergent perspective* that coordinates analyses of classroom microcultures at the social and individual levels and that emphasizes reflexive relationships between the learning of classroom communities and that of individuals. These researchers have characterized the learning of classroom communities in terms of social norms, sociomathematical norms, and classroom mathematical practices. Social norms include explaining and justifying in any domain, sociomathematical norms include what count as different mathematical solutions, and classroom mathematical practices are taken-as-shared ways of reasoning and arguing about particular mathematical ideas. McClain and Cobb analyzed emerging sociomathematical norms in one classroom, and Bowers et al., Cobb, and Cobb et al. analyzed emerging classroom mathematical practices in other classrooms. Practices become taken-as-shared when they are beyond justification (Bowers et al., 1999; Cobb et al., 2001) and can include uses of representations (see Cobb, 1999, p. 28). On balance, these researchers have focused their analyses more closely on the learning of classroom communities than on the learning of individuals, which they characterize in terms of beliefs and understandings that are psychological correlates of norms and practices.

In related work, Hall and Rubin (1998) analyzed the reflexive development of classroom practices and individual students' understandings for representing rates over four lessons. Although Sfard (2000) did not emphasize mathematical practices in her analysis of classroom discourse, her delineation of pronounced, attended, and intended foci can be understood as an analysis of salient features of representations and situations becoming taken-as-shared within a single lesson. Lobato et al. (2003) introduced the construct of focusing phenomena, which are observable features of the classroom environment that direct students' attention and that emerge through coconstructed mathematical language, features of curricular materials, and uses of artifacts.

This study extends past research (a) by examining an approach to two-digit multiplication based on areas of rectangles understood as arrays of unit squares, a context that past research suggests can be accessible to fourth-grade students, (b) by extending to research on multiplication a fine-grained perspective on representations and problem solving that has been more prevalent in research on algebra, and (c) by coordinating an analysis of evolving classroom practices over the course of an entire instructional unit with an analysis of student problem solving at the end of the unit. The classroom teacher in this study did not focus on the same norms for justifying solutions as did teachers at the center of previous studies that have used the emergent perspective. In turn, this led to possibly different standards for classroom mathematical practices becoming taken-as-shared. The contrast will lead to hypotheses discussed at the end of the article, and to be pursued in future research, about relationships between norms, how practices emerge, and consequences for individual students' learning.

REPRESENTATIONS AND PROBLEM SOLVING IN THE CHILDREN'S MATH WORLDS TWO-DIGIT MULTIPLICATION UNIT

This study was conducted in the context of the Children's Math Worlds project (CMW), an ongoing project that develops instructional materials for elementary school mathematics and conducts research on teaching and learning as teachers use those materials in their classrooms. A main objective of CMW is to make the goals of the *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) accessible to urban and suburban students and teachers. Relevant to this study are standards about number and operations, geometry, representation, measurement, and problem solving.

The CMW two-digit multiplication unit builds on a preceding CMW unit that develops connections among single-digit multiplication, equal groups, and areas of rectangles understood as arrays of unit squares. The single-digit materials include rectangles with drawn unit squares and ask, "What is the total number?" To take one example, students can calculate the area of a three-by-five rectangle by counting 15 individual unit squares, adding three groups of five unit squares, adding five groups of three unit squares, or recalling $3 \times 5 = 15$. The two-digit multiplication unit builds on connections developed in the single-digit unit, and students discover that effective single-digit strategies, like those just described, become inefficient with larger factors.

The two-digit multiplication unit scaffolds the development of more efficient strategies with three area representations and a numeric method that break apart factors and products into smaller, easier-to-handle pieces (see Figure 1). *Unit squares* representations show all unit squares and are a continuation of area representations used in the single-digit unit. The two-digit unit introduces *100s/10s/1s* and *quadrants* representations. The *100s/10s/1s* representations break apart lengths into individual tens and ones and areas into groups of 100, 10, and individual unit squares. Groups that are 10 unit squares wide and long are called *100 squares*. The *100s/10s/1s* representations help students determine magnitudes of partial products, the first coordination discussed previously.

Unit squares and *100s/10s/1s* representations are drawn to the same scale, but *quadrants* representations are not. *Quadrants* representations are intended as sketches to be used with numbers of any size and scaffold connections between expanded forms and the distributive property, the second coordination discussed previously. Finally, each line of the *expanded algorithm* corresponds to one region in *quadrants* representations. The expanded algorithm generalizes to any number of digits and can collapse to the traditional U.S. method, but represents partial products and multidigit addition more explicitly. As the unit progresses to larger factors, first unit squares and then *100s/10s/1s* representations are dropped both for reasons of scale and to transition to numeric methods.

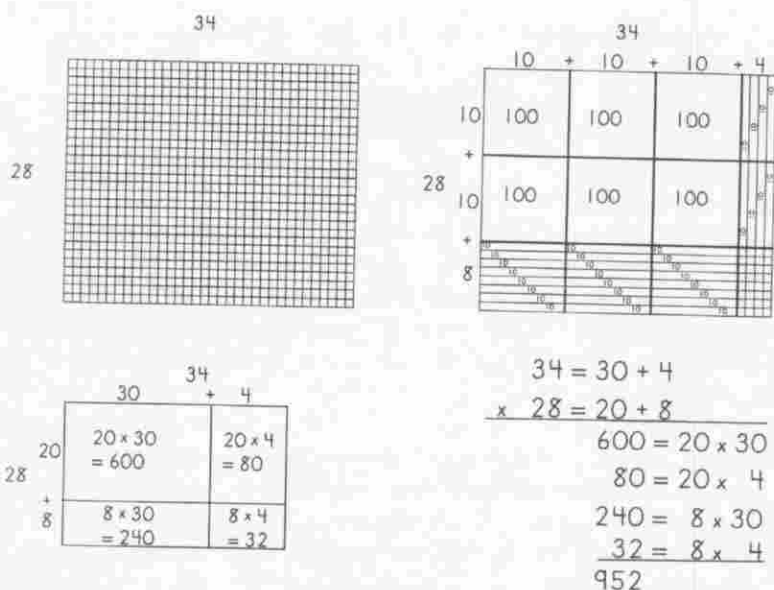


FIGURE 1 The unit squares, 100s/10s/1s, quadrants, and expanded algorithm representations for 28×34 . From "Exploring the use of new representations as a resource for teacher learning," by A. Izsák and M. Sherin, 2003, *School Science and Mathematics*, 103 (1), p.20. Copyright 2003 by School Science and Mathematics Association. Reprinted with permission.

Other two-digit multiplication materials (e.g., Manfre, Moser, Lobato, & Morrow, 1992) have used area representations and numeric methods similar to the expanded algorithm. The CMW materials are distinguished by the coordinated use of all three area representations described previously and the expanded algorithm. The unit provides particular representations and a target computation method, but also encourages teachers to elicit, discuss, and build on students' problem-solving strategies. The intent is for teachers to help students understand the area representations and master numeric methods that generalize to larger factors.¹

Three observations about solving problems with the CMW materials are key to my analysis of teaching and learning. First, teachers and students must have strategies for accomplishing the following sequence of goals: (a) find groups of (in some cases imaginary) unit squares in unit squares, 100s/10s/1s, and quadrants representations; (b) determine areas of regions found in (a); and (c) find final products by adding areas found in (b). I coordinate analyses of whole-class solutions and in-

¹Mental strategies—such as 4×17 is 4×10 plus 4×7 and $40 + 28 = 68$ —are important and rely on at least implicit use of the distributive property. Investigating student's understanding of such strategies, however, was beyond the scope of the study.

dividual student strategies by examining how one fourth-grade teacher and her students accomplished these three goals during lessons and how students accomplished the same three goals during end-of-unit interviews.

Second, teachers and students must not only connect features of each representation that correspond to the same factor or product but also understand how to use particular features to accomplish a given goal. Figure 2 illustrates four different features of unit squares representations that can be used to determine areas. Strategy (a) relies on counting groups of 10 unit squares as in "10, 20, 30, 40," whereas strategies (b), (c), and (d) rely on multiplying dimensions inscribed in three different ways. The strategy shown in Figure 2b uses the number of unit line segments along the top and side; the strategy shown in Figure 2c uses the number of unit squares. For the analyses that follow, I will take such strategies to be distinct because, as shading indicates, each relies on different representational features. Focusing on the different representational features that can be used to accomplish a given goal highlights how the same representation can be used to solve a problem in more than one way. Moreover, as research summarized previously on the learning of algebra has suggested, and as analysis of student interviews later in this article will confirm, coordinating representational features with goals can be a significant, though often underemphasized, aspect of student learning.

Third, as the CMW materials first drop unit squares and then 100s/10s/1s representations, the remaining area representations contain fewer and fewer features on which to build strategies. Thus, teachers and students who rely on unit squares for initial strategies have to refine or replace those strategies as the lessons progress.

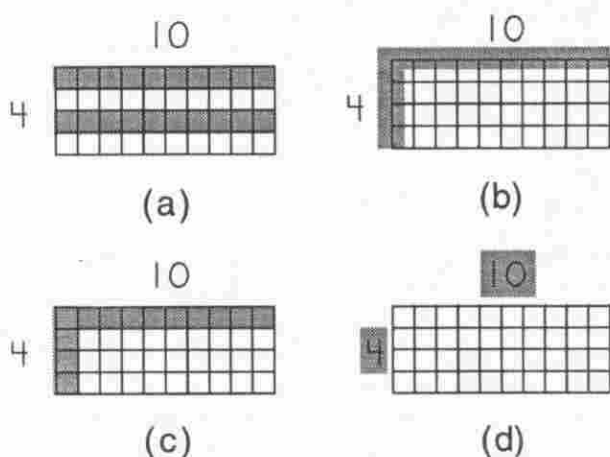


FIGURE 2 Four strategies for determining the area of a 4-by-10 rectangle using the unit squares representation: (a) counting groups of 10 unit squares, (b) multiplying the number of unit line segments along the top and side, (c) multiplying the number of unit squares along the top and side, and (d) multiplying the numeric labels.

By including representations with fewer and fewer features, the instructional materials can scaffold convergence toward multiplication strategies that coordinate magnitudes of partial products, expanded forms for factors, and the distributive property.

When examining classroom lessons, I analyze strategies, like those shown in Figure 2, as taken-as-shared means of using representational features for accomplishing problem-solving goals. Such practices emerge as students and their teacher contribute to solutions. The analysis illuminates reflexive relationships between classroom practices and individual activity consistent with the emergent perspective discussed previously, and the article will close with questions for future research within this perspective. When analyzing student interviews, I analyze strategies as knowledge that emerges as students make sense of taken-as-shared strategies or construct alternative strategies for accomplishing goals. Of course, students' existing understandings influence which features they find salient: students might attend to shaded features in Figure 2b or 2d either if they understood quantitative relationships between dimensions and area, or if they used the $l \times w = a$ formula by rote.

WHOLE-CLASS SOLUTIONS TO TWO-DIGIT MULTIPLICATION PROBLEMS

This section describes Mrs. Tate's² mathematics lessons, explains the classroom data and methods used to analyze those data, and presents an analysis of problem-solving strategies as practices constructed through whole-class discussions. The analysis will lead to the first two results of the article and will set the stage for subsequent analysis of student problem-solving strategies.

Mrs. Tate's Mathematics Lessons

The Children's Math Worlds project collaborated with Mrs. Tate over the course of the 1999-2000 school year. She was a fourth-year teacher in a midwestern suburban district with a heterogeneous population. Approximately one quarter of her students were main-streamed with learning disabilities, and about the same proportion were bilingual. Mrs. Tate was energetic and volunteered to pilot CMW materials after hearing about the project from third-grade teachers at her school. Mrs. Tate taught the CMW single-digit unit before she taught the two-digit multiplication unit. When teaching multidigit multiplication in previous years, Mrs. Tate had taught the sequence of steps in the traditional algorithm. The expanded algorithm

²All names are pseudonyms.

and using area representations to help students understand two-digit multiplication were new for her.

Mrs. Tate executed engaging, well-managed mathematics lessons. Students knew that they were to stay focused and participate. They did so by listening, watching, suggesting strategies, asking questions, and working with their classmates in groups when asked to do so. Mrs. Tate incorporated humor and dealt with interruptions in ways that minimized disruption.

Mrs. Tate usually began lessons with 15- to 20-min activities that she called "Do Nows." These activities were not written into the CMW instructional materials, but were one of Mrs. Tate's established classroom routines. A typical Do Now consisted of a single page with mathematical puzzles, exercises that reviewed a previously studied topic, or a warm-up for the day's lesson on two-digit multiplication. Mrs. Tate passed out the day's Do Now to each student and then worked with individuals and small groups. As the two-digit multiplication lessons progressed, Mrs. Tate also used this time to work with students who needed extra help on CMW homework.

Mrs. Tate spent the second half of her lessons leading whole-class solutions to one or two two-digit multiplication problems. Mrs. Tate used transparencies on the overhead projector that reproduced student pages from the CMW materials. The class discussed connections among the representations and strategies that either Mrs. Tate or students proposed. Mrs. Tate asked frequent questions to monitor the class' understanding, answered students' questions, and encouraged students to offer strategies—sometimes at the projector. Periodically, she stopped class discussions for a few minutes so that students could work on problems at their desks with neighbors.³

Data and Methods For Analyzing Whole-Class Solutions

I videotaped all 13 two-digit multiplication lessons, which were spread over a period of 8 weeks broken up by state testing and practice for this testing, Thanksgiving, and Christmas. Mrs. Tate and her students were accustomed to my presence and the video equipment, because I had previously taped several lessons from the single-digit multiplication unit. I taped the entire classroom and zoomed in on the overhead frequently to capture what Mrs. Tate and her students wrote and where they pointed as they explained strategies. Thus, with one camera I captured in as much possible detail whole-class interactions and demonstrated strategies for using area representations to solve two-digit multiplication problems.

³I use the word *discussion* to refer to talk that occurred during whole-class solutions. The extent to which Mrs. Tate was adaptive to student contributions varied within and across lessons.

I analyzed solutions demonstrated by Mrs. Tate and her students over the entire set of lessons, using a method similar to that described by Cobb and Whitenack (1996) for handling large, longitudinal sets of videorecordings. This approach is compatible with Glaser and Strauss' (1967) constant comparative method. In my first pass through the videotapes, I analyzed the goals that Mrs. Tate and her students accomplished problem-by-problem. Mrs. Tate and her students consistently found groups, determined areas, and found final products—the three goals identified previously—but they did not always articulate these goals explicitly. I then formed initial categories for the representational features that were used to accomplish each of the three, sometimes implicit, goals. I used verbal references, hand gestures, and added inscriptions (e.g., shading, underlining, and circling) as evidence for the representational features to which Mrs. Tate and her students attended. The videotapes captured verbal references and added inscriptions consistently, but sometimes failed to capture rapid hand gestures. In such cases, I noted alternative representational features to which Mrs. Tate or her students may have referred. I then refined my categories by taking subsequent passes through the videotapes until the categories became stable. A second researcher examined independently one early, one middle, and one late solution (14×13 , 25×63 , and 75×23 in Table 1) and arrived at a consistent analysis.

Table 1 summarizes the results of the analysis, and the following presentation illustrates each category of strategy listed. Note that I could construct the categories in Table 1 by examining several solutions using a given combination of area representations, because Mrs. Tate and her students solved 11 problems using unit squares, 100s/10s1s, and quadrants representations; 5 problems using just 100s/10s1s and quadrants representations; and 6 problems using only quadrants representations. Distilling strategies used to solve 22 problems over 13 lessons into one table required omission of details unique to particular solutions. However, when analyzing the end-of-unit student interviews (Results 3 and 4), I fill in such details when they appeared to shape students' problem solving.

Result 1

Taken-as-shared class strategies for finding groups and determining areas in unit squares and 100s/10s/1s representations emerged as Mrs. Tate and her students discussed several alternatives.

The class solution to 28×34 illustrates Result 1, and Figure 3 shows the instructional materials for this problem. Mrs. Tate and her students had already solved 10 problems in which factors were between 1 and 19. The problem 28×34 was the first in which factors were greater than 20 and the last for which materials included

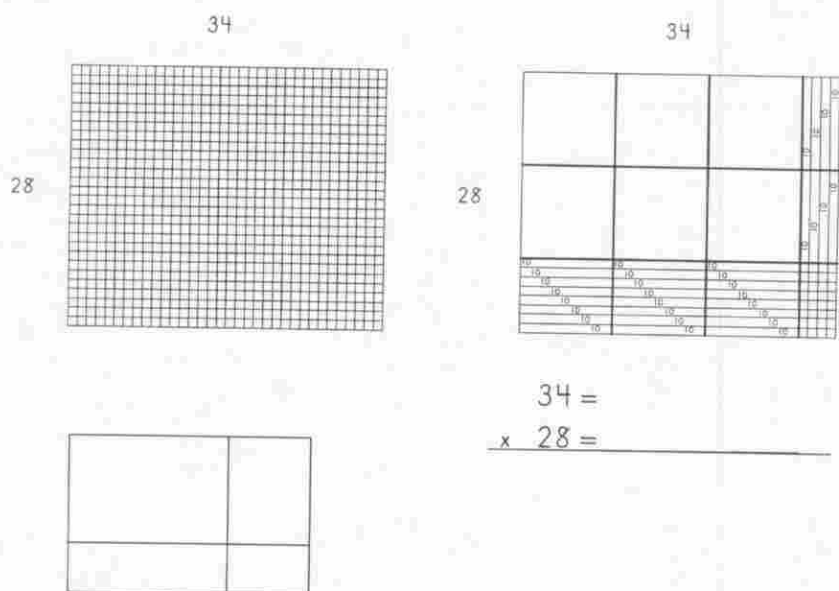
TABLE 1
Class Strategies for Finding Groups, Determining Areas,
and Finding Final Products

<i>(Area Reps)</i> <i>Problems</i>	<i>Find Groups</i>	<i>Determine Areas</i>	<i>Find Final Products</i>
(U,H,Q) ^a			
10 x 13 ^b	Counted 10 unit squares across and down	TT ^c Counted areas by 10s ^d Stated 10 x 10 = 100	Copied partial products to complete expanded algorithm ^e
14 x 13		OT Counted areas by 10s	
18 x 16	Added 10 + 1s labels	TO Counted areas by 10s	
17 x 16		OO Counted areas by rows	
13 x 17		Counted unit squares across and down and multiplied	
12 x 18	Drew lines from plus signs		
18 x 13			
10 x 16			
4 x 17			
9 x 18			
28 x 34			
(-,H,Q)			
27 x 23	Broke factors into single 10s and 1s directly	TT Counted 100 squares Summed 10s in labels ^f	Copied partial products to complete expanded algorithm
36 x 28		OT Counted perimeter squares by 10s Summed 10s in labels	
25 x 63	Drew lines from plus signs	Pointed to 1s in labels	
66 x 48		TO Counted perimeter squares by 10s Summed 10s in labels Pointed to 1s in labels	
33 x 47		OO Pointed to 1s in labels	
(--,Q)			
62 x 47	Wrote factors in expanded form	TT Pointed to 10s in labels	Copied partial products to complete expanded algorithm
75 x 23		TO Pointed to 10s and 1s in labels	
12 x 48		OT Pointed to 10s and 1s in labels	
52 x 36		OO Pointed to 1s in labels	
15 x 49			
99 x 28			

^aU = unit squares, H = 100s/10s/1s, and Q = quadrants representations. ^bThe first factor was represented vertically, the second horizontally. ^cTT = tens x tens (top left quadrant), TO = tens x ones (top right quadrant), OT = ones x tens (bottom left quadrant), OO = ones x ones (bottom right quadrant).

^dSee Figure 2a for an example. ^eThroughout the two-digit multiplication unit, Mrs. Tate and her students used quadrants representations to record partial products, which they then used to complete the expanded algorithm. They did not explicitly discuss quadrants as representations of areas in unit squares and 100s/10s/1s representations. The CMW materials did not emphasize sufficiently that quadrants representations were to be understood as sketches of unit squares and 100s/10s/1s representations.

^fThe remaining descriptions in the column "Determine Areas" are of strategies for finding dimensions, which were then multiplied.

FIGURE 3 Instructional materials for 28×34 .

a unit squares representation. Thus, analysis of this solution will illustrate the entries for the first group of problems in Table 1 (10×13 through 28×34) and many of the entries for the second group (27×23 through 33×47). Mrs. Tate focused the lesson on finding groups of 100, 10, and individual unit squares in the unit squares representation, determining areas of the resulting regions, recording calculations in the quadrants representation, and completing the expanded algorithm. She did refer to the 100s/10s/1s representation provided in the materials, but only in passing. Perhaps this was because she constructed the 100s/10s/1s representation on top of the unit squares representation.

The transition to larger factors and fewer area representations introduced two mathematical issues into whole-class discussions. First, the class had to extend to larger factors existing taken-as-shared strategies developed over previous solutions for finding groups and determining areas. Such extensions were not straight forward because the larger factors introduced multiple 100 squares. Thus, when determining the area of the 20-by-30 region, students could take 100 squares, rows of unit squares, or columns of unit squares as the repeated group. The class focused on 100 squares at first, but coordinating expanded forms for factors and the distributive property required focusing instead on the dimensions of the 20-by-30 region. As a result, Mrs. Tate had to redirect students' attention toward representational features that corresponded to these dimensions. Second, because the instructional

materials were about to drop unit squares representations, the class had to develop new strategies that relied on numeric labels. Mrs. Tate discussed connections between strategies that relied on unit squares and strategies that relied on numeric labels, but subsequent analysis of interview data will show that not all students made similar connections.

Goal 1: Find Groups

The unit squares representation. Mrs. Tate began the lesson by discussing the following problem included in the instructional materials:

A sandwich shop delivers to houses and businesses in a part of Chicago that is 28 blocks long and 34 blocks wide. Over how many square blocks does the sandwich shop deliver?

She explained that Chicago streets often form a grid and asked her class, "What would the equation be that would solve this problem?" After several students suggested 34×28 , Mrs. Tate turned the class' attention toward the unit squares representation and asked, "What are you going to do with this problem? This is a lot bigger than we are used to!" She gave students 5 min to work on the problem using a 28-by-34 unit squares representation at their desks and, as she circulated, asked questions like, "What are you breaking up your rectangle into?"

Mrs. Tate reconvened the class and asked for ideas.⁴ Rachel⁵ suggested using 4 groups of 7 by 34, but could not figure out 7×34 . Other students suggested groups of 100s, 10s, and 1s, and one student began finding 100 squares on the overhead. The student got stuck after finding one 100 square, and comments by other students made clear that many were uncertain of the number of 100 squares in the problem. Mrs. Tate reiterated students' suggestions to use groups of 100s, 10s, and 1s and reminded the class that in a previous lesson they had broken apart 18 into $10 + 8$.

Mrs. Tate rephrased one student's "three times 10" suggestion, saying "three 10s and a 4." She counted unit squares across the top row starting at the left-hand end. She left a visible dot in each square as she counted out loud "1, 2, 3, 4, 5, 6, 7, 8, 9, 10," wrote "+" over the border between the 10th and 11th squares, and wrote "10" over the approximate center of the squares that she had counted. Mrs. Tate repeated twice more the process of counting to 10, marking squares with dots, and labeling the top of the rectangle. For the final group, she

⁴As the class solved the problem, Mrs. Tate responded to a complex set of mathematical issues raised by students, and the original sandwich shop context dropped out of discussion.

⁵I use proper nouns for students who also participated in the end-of-unit interviews.

left no dots as she counted squares and wrote "4." She then demonstrated the analogous procedure down the left-hand edge of the unit squares representation. Figure 4a shows Mrs. Tate's unit squares representation at this point. Light gray indicates initial representational features that Mrs. Tate used, and dark gray indicates inscriptions that she added.

Mrs. Tate then told her students that the goal was an "efficient" solution. As she said this, she drew vertical and horizontal lines that began by the plus signs in her numeric labels and went across the unit squares representation. Although Mrs. Tate did not discuss positions of plus signs explicitly, she apparently used these to locate the lines that she drew. Figure 4b shows that Mrs. Tate's lines divided unit squares into 6 groups of 100, 2 groups of 40, 3 groups of 80, and 1 group of 32.

Analysis. All proposed strategies, including the 4 groups of 7 by 34 strategy, suggested that students attended to connections between multiplication and equal groups but were unsure how to use such connections in problems with larger factors. Students who suggested finding groups of 100s, 10s, and 1s apparently were trying to extend strategies that the class had used to solve 10 problems during previous lessons. The column labeled *Find Groups* in Table 1 shows that the class had counted groups of 10 unit squares along perimeters of unit squares representations ("Counted 10 unit squares across and down"); had introduced numeric labels in which plus signs marked the first 10 unit squares ("Added 10 + 1s labels"); and had used those plus signs to locate and draw lines that created groups of 100, 10, and individual unit squares ("Drew lines from plus signs"). The "three times ten" suggestion did not make explicit how to extend these strategies to larger factors, but Mrs. Tate did so.

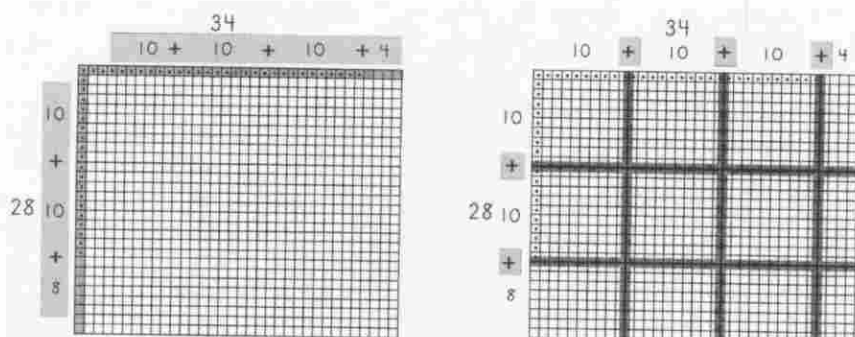


FIGURE 4 Mrs. Tate found groups by (a) counting unit squares to construct additional numeric labels and (b) using plus signs to locate vertical and horizontal lines.

Goal 2: Determine Areas

The 20-by-30 region (Tens by Tens [TT]). To start the class determining areas of regions in the unit squares representation, Mrs. Tate wrote "100" in each 100 square and asked, "Why are these 100 squares where they are? Why are they here? Why aren't the 100 squares down here (she pointed to the 8-by-10 regions)?" One student explained that there were "not enough" unit squares in the 8-by-10 regions. When Mrs. Tate asked if there was "a better way you could represent [the six] hundreds," another student proposed calculating 100×6 . Mrs. Tate was out of time for the day.

When the lesson resumed the next day, students continued to focus on the 100 squares in the 20-by-30 region as shown in Figure 5a. For example, Pete offered the following strategy, "I add up the 100s, and then I added the three 40s on the side, and then I add up the three 80s on the bottom. By "three 40s" Pete may have meant the two 10-by-4 regions and the 8-by-4 region, but Mrs. Tate did not pursue Pete's error. Instead, she stayed focused on the 20-by-30 region and told students that they were looking for "a more efficient equation." She reminded students that they had used 10s in factors for 18×13 to write " $10 \times 10 = 100$ " and underlined the 30 in "34" and the 20 in "28." She then asked, "What kind of multiplication problem could you put together to give us 600, Rachel knows, using that 3 and 2?" Rachel suggested 30×20 but did not connect her approach to area representations. Many other students raised their hands to show agreement. (Because Rachel was a

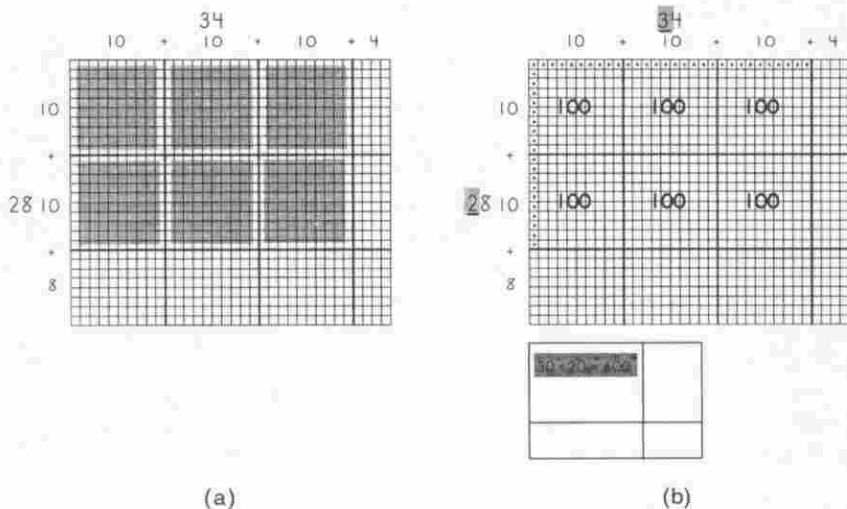


FIGURE 5 Two strategies for determining the area of the 20-by-30 region: (a) Students counted 100 squares (100×6) and (b) Mrs. Tate found dimensions in numeric labels (30×20).

strong student, other students may have gone along without fully understanding her suggestion.)

Having explicitly directed students' attention toward the 20 and 30 in the numeric labels, Mrs. Tate then introduced a second strategy for finding dimensions: "Up here there are 10, 20, 30 (she pointed toward the unit squares representation, but her precise gestures were off camera). Three tens equals how many?" The class chorused, "30." Mrs. Tate then asked how much two 10s made (her gestures toward the unit squares representation were off camera again). She called on a student who answered, "20." Mrs. Tate wrote " $30 \times 20 = 600$ " in the corresponding region of the quadrants representation, traced the perimeter of the 20-by-30 region in the unit squares representation, and told students that 600 was "the area of this big, big rectangle." Figure 5b shows what Mrs. Tate's unit squares and quadrants representations looked like at this point. The figure does not show Mrs. Tate's second strategy because, without the hand gestures, I could not determine the precise representational features to which she referred.

Analysis. Mrs. Tate and her students discussed strategies for accomplishing a new goal: determining areas of several 100 squares combined. Student comments, such as there were "not enough" unit squares for the 8-by-10 regions to be 100 squares, suggested that at least some students focused on areas of regions. Students who proposed calculating 100×6 and adding six 100 squares may have extended the strategy of counting repeated groups of 10 unit squares used in previous class solutions ("Counted areas by 10s" in Table 1 and Figure 2a), but this strategy did not coordinate magnitudes of partial products, expanded forms for factors, and the distributive property.

Mrs. Tate redirected students' attention by offering two new strategies for finding the dimensions of all six 100 squares combined. The first relied on the 30 and 20 in the "34" and "28" labels. Mrs. Tate had used a similar strategy only once before during the solution to 18×13 . The second strategy involved counting by 10s to determine dimensions. Without Mrs. Tate's hand gestures, I could not tell whether her second strategy relied on her " $10 + 10 + 10 + 4$ " and " $10 + 10 + 8$ " labels or on unit squares along the perimeter of the 20-by-30 region. In subsequent solutions, Mrs. Tate used both features, sometimes summing 10s in numeric labels ("Summed 10s in labels" in Table 1) and sometimes counting groups of 10 imaginary unit squares along perimeters ("Counted perimeter squares by 10s" in Table 1). Figures 2a and 2c show that the strategies "Counted areas by 10s" and "Counted perimeter squares by 10s" relied on distinct, albeit overlapping, representational features.

The 8-by-30 region (Ones by Tens [OT]). The class discussed strategies for the 8-by-30 region next. One student suggested "30 times 8." When Mrs. Tate asked where the 30 came from, the student replied, "The three tens." Mrs. Tate

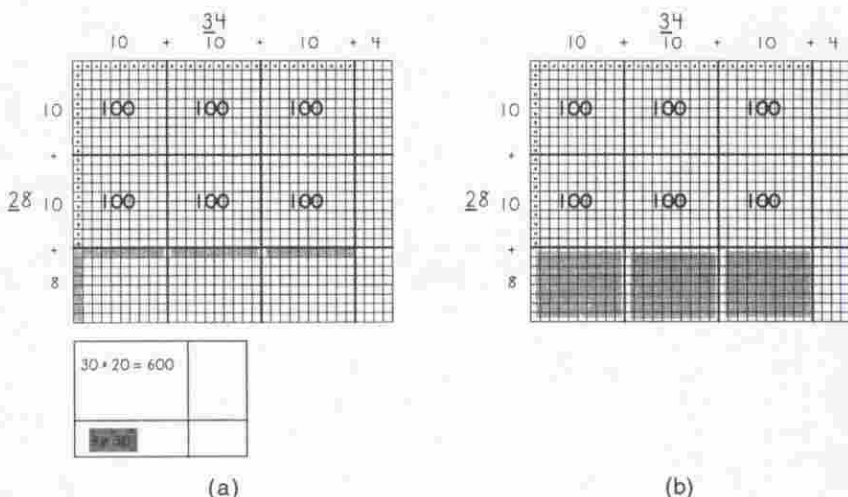


FIGURE 6 Two strategies for determining the area of the 8-by-30 region. (a). Using unit squares along the perimeter (8×30). (b). Using three groups of 80 unit squares (80×3).

elaborated the explanation, saying "The three 10s are 30 (she pointed to each of the 8-by-10 regions). So you are having 30×8 . OK. 10 across the top, 8 down (she traced the top and left-hand edges of the left most 8-by-10 region)." Maria suggested 80×3 , apparently focusing on three groups of 80 unit squares. Mrs. Tate responded, "Truthfully you could do it either way, 30×8 or 80×3 ." She explained that she was going to do 30×8 because that was the way she had her calculations on her paper.⁶ Mrs. Tate wrote " 8×30 " in the corresponding region of the quadrants representation and pointed to each 8-by-10 region again as she counted, "10, 20, 30." Figure 6a shows representational features for the 8×30 strategy and what Mrs. Tate's unit squares and quadrants representations looked like at this point. Later she would add " $= 240$ " in the quadrants representation. Figure 6b shows representational features for Maria's 80×3 strategy.

Analysis. The "30 times 8" proposal and subsequent "three tens" explanation suggested that some students focused on the dimensions of the three 8-by-10 regions combined. The representational features to which these students attended and the features that Mrs. Tate emphasized, however, may or may not have been the same. Student comments did not specify features, but Mrs. Tate clearly counted

⁶The CMW project did not anticipate how hard it would be for students to focus on dimensions of several regions combined. Mrs. Tate's response to unexpected student ideas should be understood as a natural part of using an instructional approach for the first time.

unit squares along the perimeter ("Counted perimeter squares by 10s"). This strategy resembled the second strategy that Mrs. Tate discussed before for the 20-by-30 region and emerged through contributions made by students and their teacher. Maria's alternative 80×3 proposal was similar to that made by Pete during the previous section ("I add up the three 80s on the bottom.") and made clear that some students focused on equal groups when determining areas, but not in ways that coordinated dimensions of combined regions. When Mrs. Tate acknowledged that 80×3 would also work, she diverted her focus from dimensions momentarily. Perhaps she wanted to validate other students' reasonable alternative strategies.

When solving problems during previous lessons in which both factors were less than 20, Mrs. Tate's class had also discussed several different equal groups, but then converged on groups of 100, 10, and individual unit squares. By the beginning of the present lesson, students did not have to justify why they were focusing on such groups. Thus, using groups of 100, 10, and individual unit squares had become beyond justification, the criterion for taken-as-shared used by Bowers et al. (1999) and by Cobb et al. (2001). The discussions that Mrs. Tate and her students had about the 20-by-30 and 8-by-30 regions would form the basis for new taken-as-shared strategies for problems with larger factors.

The 20-by-4 and 8-by-4 regions (Tens by Ones [TO] and Ones by Ones [OO]). Mrs. Tate sought fewer student suggestions for the 20-by-4 and 8-by-4 regions and may have been concerned with finishing the solution in the remaining time for the day's lesson. She reestablished her focus on dimensions by counting unit squares along perimeters. She asked how many "little blocks" were in the 20-by-4 region and pointed to each unit square across the top as she counted, "1, 2, 3, 4 going this way." She then asked "How many rows?" and pointed to each unit square in the right-most column as she counted, "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. What's 10 and 10?" (Mrs. Tate may have intended connections to the "Counted perimeter squares by 10s" strategy that the class had constructed for the 8-by-30 region.) Several students said "20," and Mrs. Tate wrote " $4 \times 20 =$ " in the corresponding region of the quadrants representation. She never completed this equation. Figure 7a shows what Mrs. Tate's unit squares and quadrants representations looked like at this point.

Finally, Mrs. Tate traced the perimeter of the 8-by-4 region in the unit squares representation and counted unit squares along the top, leaving visible dots as she went. "There are four blocks across the top of this rectangle. I know this because there is a 4 up here (she circled the 4 in her " $10 + 10 + 10 + 4$ " label)." Mrs. Tate then asked, "Four blocks times how many rows?" To answer this question, she circled the 8 in her " $10 + 10 + 8$ " label and drew an arrow to the 8-by-4 region, as shown in Figure 7b. She said, "There are 8 blocks going down on this column (she traced the left-hand edge of the 8-by-30 region) so there must be 8 on this side (she traced the right-hand edge of the 8-by-4 region)." Mrs. Tate concluded by writing

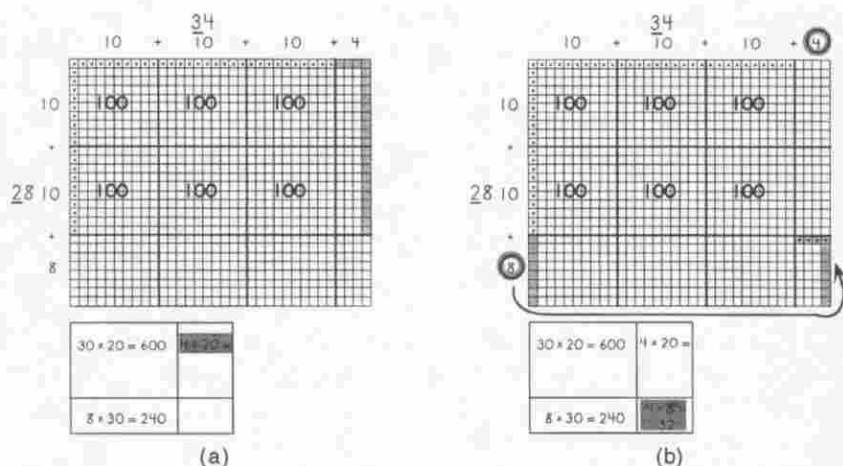


FIGURE 7 (a). Mrs. Tate counted unit squares to find dimensions of the 20-by-4 region. (b). Mrs. Tate counted unit squares and pointed to labels to find dimensions of the 8-by-4 region.

" $4 \times 8 = 32$ " in the corresponding region of the quadrants representation. Figure 7b shows Mrs. Tate's unit squares and quadrants representations at this point.

Mrs. Tate accomplished the third goal, find final products, by copying partial products from the quadrants representation to the expanded algorithm. She asked the class, "Now what do you do with these numbers?" The class chorused, "Add 'em."

Analysis. In addition to refocusing the class discussion on dimensions by counting unit squares, Mrs. Tate also connected numeric labels to unit squares when finding dimensions of the 8-by-4 region. The arrow that she drew explicitly inscribed the connection between her " $10 + 10 + 8$ " label and unit squares along the vertical dimension of the 8-by-4 region. Such use of labels was similar to the strategy she demonstrated when underlining the 20 and 30 and connected new strategies that relied on numeric labels to previous strategies that relied on unit squares. Table 1 shows that this "Pointed to 1s in labels" strategy was continually used in subsequent solutions.

Summary

In solving 28×34 , Mrs. Tate's class extended existing taken-as-shared strategies for finding groups and determining areas to problems with larger factors. The range of strategies afforded by unit squares and 100s/10s/1s representations provided teaching and learning opportunities, because students who had connected multiplication, equal groups, and areas of rectangles to different degrees

could discuss alternative strategies. The same range of strategies also created teaching challenges, because some students extended existing taken-as-shared strategies in ways that could coordinate magnitudes of partial products, expanded forms for factors, and the distributive property, others did not (e.g., 100×6 for the 20-by-30 region and 80×3 for the 8-by-30 region). Mrs. Tate directed students' attention toward dimensions of rectangles and connected new methods for determining areas that relied on numeric labels to previously established methods that relied on unit squares. Thus, under Mrs. Tate's guidance, student suggestions formed the basis for new class strategies that, in turn, would become beyond justification, and hence taken-as-shared. This reflexive relationship between learning of the classroom community and that of individuals is consistent with the *emergent perspective*. Note that Bowers et al. (1999), Cobb (1999), and Cobb et al. (2001) also have documented the emergence of classroom mathematical practices over the course of a small number of classroom discussions. Result 2 will show that in subsequent lessons class strategies continued to converge on those that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property. Result 4 from student interviews discussed later in the article reveal further aspects of understanding area representations that did not surface in whole-class discussions.

Result 2

Subsequent class strategies converged on those that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property.

The second and third groups of entries in Table 1 (starting with 27×23 and 62×47 , respectively) summarize class strategies over the rest of Mrs. Tate's two-digit multiplication lessons. The second group of problems were those for which the CMW materials provided only 100s/10s/1s and quadrants representations. The third group was those for which students had to draw their own quadrants representations. The convergence of class strategies on those that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property was consistent with the goals of the CMW two-digit multiplication unit. The following summary of data and analysis that led to Result 2 will show that several key features of the 28×34 solution were not isolated: New class strategies continued to emerge through interactions during which Mrs. Tate solicited students' ideas, directed students' attention toward dimensions, and connected strategies that relied on numeric labels to those that relied on unit squares.

As factors got larger, and the CMW materials first dropped unit squares and then 100s/10s/1s representations, counting unit squares was no longer feasible for finding groups. Mrs. Tate introduced the new strategy of breaking apart factors di-

rectly into sums of single 10s and 1s. Figure 8 shows how Mrs. Tate broke apart the factors 63 and 25 into $10 + 10 + 10 + 10 + 10 + 10 + 3$ and $10 + 10 + 5$ ("Broke factors into single 10s and 1s directly" in Table 1). Then Mrs. Tate drew vertical and horizontal lines from plus signs as she had done in previous solutions. In this one solution, Mrs. Tate also left a visible trace of dots, shown in Figure 8, that she said "symbolize[d] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 little blocks." This demonstration connected the previous class strategy for finding groups, which relied on unit squares, to the new strategy, which relied on numeric labels.

To determine the area of the 20-by-60 region, Mrs. Tate used single 10s in her numeric labels to determine the area of each 100 square. She then summed the 10s across and down to state the corresponding 20×60 multiplication problem ("Summed 10s in labels" in Table 1) and counted the 100 squares to determine the total area of 1,200 unit squares ("Counted 100 squares" in Table 1). In this way, she built on past student suggestions to count 100 squares and coordinated magnitudes of partial products, expanded forms for factors, and the distributive property. For the 5-by-60 region, Mrs. Tate counted five imaginary unit squares down, summed the six 10s across, and multiplied. For the 20-by-3 region, she pointed to the "3," summed the two 10s down, and multiplied. For the 5-by-3 region, she pointed to the labels and multiplied. The class continued to find final products by copying partial products from quadrants representations to the expanded algorithm and adding.

The final problems in the CMW two-digit multiplication unit involved products greater than 1,000. The materials provided factors only, and students had to sketch their own quadrants representations. As Mrs. Tate circulated one day, she saw Maria adding numeric labels to her quadrants representation by writing factors in expanded form across the top and down the left-hand side (similar to the quadrants representation shown in Figure 1). Mrs. Tate had never added numeric labels to

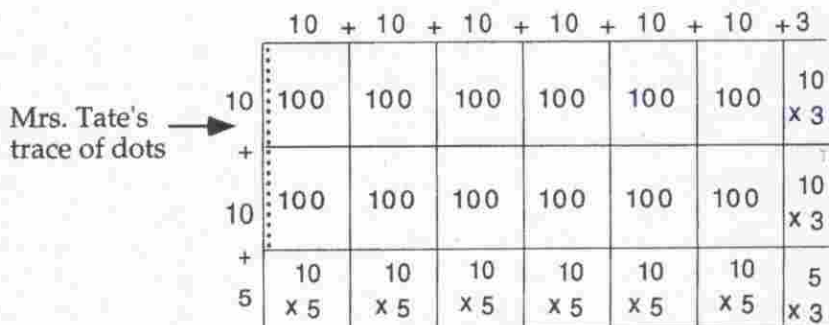


FIGURE 8 Mrs. Tate's 100s/10s/1s representation for 25 times 63.

quadrants representations, but now showed Maria's work to the whole class and used similar labels ("Wrote factors in expanded form" in Table 1) to determine partial products ("Pointed to 10s in labels," "Pointed to 10s and 1s in labels," and "Pointed to 1s in labels" in Table 1). Working subsequent problems at their desks, most students added similar labels, determined areas by multiplying 10s and 1s in these labels, and found final products by copying and adding partial products to complete the expanded algorithm. Thus, contributions made by Mrs. Tate and her students led to class strategies that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property.

Summary

Table 1 summarizes the evolution of taken-as-shared class strategies for finding groups, determining areas, and finding final products. Mrs. Tate and her students discussed a range of strategies for using unit squares representations, but this range decreased as the CMW materials dropped unit squares and 100s/10s/1s representations. Analyses of solutions to 28×34 and 25×63 show how Mrs. Tate continued to build on students' suggestions and to connect new strategies that relied on numeric labels to old strategies that relied on unit squares. Ultimately, class discussions converged on essentially one multiplication strategy that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property. As a consequence of this convergence, alternative choices for equal groups, like those that students suggested during the solution to 28×34 , faded from class discussions.

STUDENT SOLUTIONS TO TWO-DIGIT MULTIPLICATION PROBLEMS

Classroom data discussed previously evidenced a range of student strategies for accomplishing problem-solving goals, but the extent to which Mrs. Tate's students appropriated taken-as-shared class strategies, or used alternative strategies, remains an important question. To gain access to students' strategies for finding groups, determining areas, and finding final products, I conducted end-of-unit interviews with pairs of Mrs. Tate's students. This section of the article describes the interview data, explains the methods used to analyze those data, and presents an analysis of student strategies. The analysis will lead to the third and fourth main results of the article and will reveal further challenges when learning two-digit multiplication with the CMW materials that did not surface in whole-class discussions. These challenges have implications discussed at the end of the article for other instructional approaches that rely on representations with multiple features.

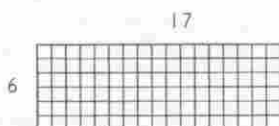
Data and Methods for Analyzing Student Strategies

To gain access to student solutions to two-digit multiplication problems, I interviewed 6 pairs of students, or about half of Mrs. Tate's class. Mrs. Tate helped identify pairs of low-, mid-, and high-achieving students who had experience working together. Four pairs consisted of students who were mid- to high-achieving. Students in one of the two low-achieving pairs received services for learning disabilities in mathematics. Two pairs were boys and 4 were girls. I used pairs of students to get more detailed access to their thinking as they compared each other's work.

The 45- to 50-min semistructured interviews (Bernard, 1994, Chapter 10) took place in an empty classroom at times when Mrs. Tate agreed to have students pulled from her class. With one exception, all interviews occurred in the final week of the two-digit multiplication unit. During the interviews, I had students solve multiplication problems like those they worked on in class and for homework. Figure 9 shows the two interview tasks selected for this analysis. The first asked stu-

Task 1

Calculate the area of the rectangle using the method from class.



Task 2

A drug store chain has 26 stores in each of 38 states. How many stores does the chain have in all?

- (1) Label the sides of the quadrant box appropriately.
- (2) Fill in the box by calculating the area of each quadrant.
- (3) Solve the equation using the method from class.

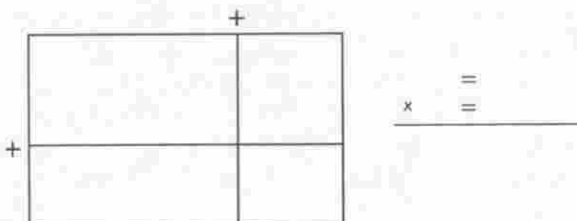


FIGURE 9 The two interview tasks.

dents to solve 6×17 using the unit squares representation, the second to solve a word problem using the quadrants representation. As one might expect, given the directions for the two tasks, students tried to produce strategies like those discussed in class. When students were either done with a problem or stuck, I asked them to explain what they had done and any difficulties they were having. Students were encouraged to ask one other questions.

I recorded the interviews using two video cameras, one to capture the students and one to capture what they wrote. I transcribed the interviews in their entirety and added notes indicating what students wrote and what hand gestures they used. I also kept all of the students' written work in case the videotapes did not capture important aspects clearly. I used microgenetic methods (Schoenfeld et al., 1993) to analyze line-by-line utterances, hand gestures, and evolving written work when determining how students used representational features to accomplish problem-solving goals. In contrast to the lesson analysis, during which I searched for patterns in observable behaviors, the intent of the interview analysis was to attribute understandings to students. Examining utterances, hand gestures, and evolving written work together allowed me to test attributed understandings more thoroughly than would have been possible if I had analyzed students' explanations or written work alone. A second researcher checked independently the analysis of interviews with Pete and Jodi.

Result 3

Four students completed both tasks correctly. All used methods that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property; 2 also used variations of class strategies that were still based on correct connections among numeric labels, unit squares, and areas of rectangles.

To complete Task 1, Rachel, Alice, and Nina found groups using the "Counted 10 unit squares across and down," "Added $10 + 1$ s labels," and "Drew lines from plus signs" strategies used in the first 11 class solutions and summarized in Table 1. All 3 students determined areas correctly, and Rachel and Nina explained their calculations in terms of dimensions found by counting unit squares along the top and side (see Figure 2c). Finally, all 3 found final products by adding the two partial products.

Figure 10 shows that Susy's solution to Task 1, though based on different groups of unit squares, was still constrained by correct connections among numeric labels, unit squares, and areas of rectangles.⁷ (A second student, Jodi, pro-

⁷I could not scan Susy's work due to multiple layers of erasures and inscriptions, so I used the videotape to reconstruct her inscriptions.

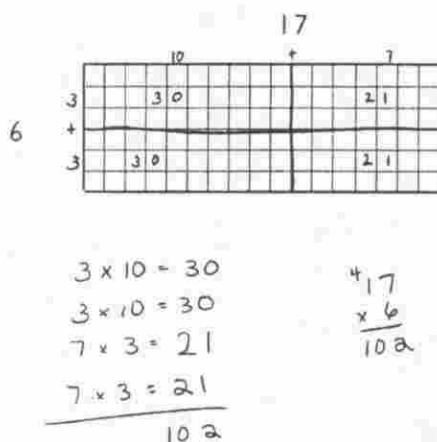


FIGURE 10 Susy's correct solution to Task 1.

duced a similar solution.) Susy found final products by adding the partial products and checked her answer using the traditional algorithm.⁸ Susy may have based her solution on one past class discussion during which students proposed and discussed strategies similar to breaking 6 into 3 + 3, but she also appeared to base her solution on strategic knowledge: She explained that smaller numbers meant she did not have “to count so much” and were easier to add.

Nina and Susy coordinated magnitudes of partial products, expanded forms for factors, and the distributive property when using the quadrants representations to complete Task 2. Alice, however, used strategies that relied on 100 squares and other regions analogous to those in Figure 8. All 3 students pointed to numeric labels when explaining how many imaginary unit squares were along the edges and how many imaginary unit squares were contained inside regions of their representations. These explanations evidenced underlying connections among numeric labels, unit squares, and areas of rectangles.

Rachel's nearly correct solution to Task 2, shown in Figure 11, occurred before labeling quadrants representations with expanded forms was discussed in class. Rachel added the “20 + 6” and “30 + 8” labels; calculated $20 \times 30 = 500$, $30 \times 6 = 180$, and $20 \times 8 = 160$; and wrote “500,” “180,” “160,” and “48” in the quadrants representation. She computed 340 and 548 mentally and summed to get 888. Rachel pointed to her labels when explaining how many unit squares were along the edges of the 20-by-30 region and at that point caught her $20 \times 30 = 500$ error. She

⁸Susy may have learned the traditional algorithm at home or from a classmate because some students brought it into Mrs. Tate's classroom. Mrs. Tate allowed students to check their work with the algorithm.

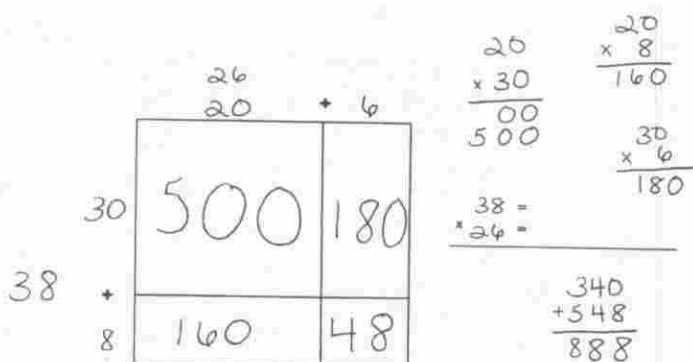


FIGURE 11 Rachel's nearly correct solution to Task 2.

corrected both the partial product and her final sum. Like Nina, Susy, and Alice, Rachel evidenced underlying connections among numeric labels, unit squares, and areas of rectangles when explaining her solution.⁹

Summary

Given the extensive literature on secondary students' difficulties with a wide range of representations, Rachel, Alice, Nina, and Susy's correct solutions to both tasks are significant because they constitute an existence proof that these fourth-grade students could solve complex problems using multiple representations. Students' explanations even for Task 2 in terms of unit squares suggested that their solutions were not rote. Susy provided vivid evidence that she understood underlying connections among numeric labels, unit squares, and areas of rectangles by varying class strategies in ways that reflected at least implicit understandings of the distributive property when factors and products were represented geometrically.

Result 4

Four other students completed Task 1 correctly and articulated correct connections among numeric labels, unit squares, and areas of regions, but struggled in Task 2 to coordinate representational features with the goal of determining areas of quadrants.

⁹Because Maria was Rachel's partner, this discussion may explain Maria's solution to 62×47 that Mrs. Tate then picked up and used with her class (see Result 2).

Pete and Jodi's work on Task 2 revealed additional challenges students faced when learning two-digit multiplication using the CMW materials that were not evident from whole-class discussions. (Maria and Rachel, Jodi and Alice, and Pete and Tom were partners.)

Pete's Work on Task 2

Summary of Pete's initial work. Pete read the drug store problem, wrote " $26 = 20 + 6$ " and " $38 = 30 + 8$ " in the space provided for the expanded algorithm, and calculated $60 + 48 = 108$. Pete may have multiplied the tens numbers to get 60, multiplied the ones numbers to get 48, and added the results. Such computations would have combined two errors that Mrs. Tate had discussed with her class: erring on magnitudes of products like 20×30 and omitting products of tens and ones.

Pete wrote " $30 + 8$ " across the top of the quadrants representation and " $20 + 6$ " down the left-hand side but then stopped, saying that he did not "know how to do this." Although Pete could not determine areas initially, he could recall the goals of finding groups, determining areas, and finding final products.

When I returned Pete to his correct solution to Task 1, shown in Figure 12, he explained three strategies for determining areas: counting all unit squares, multiplying dimensions found in numeric labels (see Figure 2d), and multiplying dimensions found in unit squares along edges of regions (see Figure 2c).

Pete's explanations suggested that he had appropriated the class strategies listed in Table 1 of "Pointed to 10s and 1s in labels" and "Counted unit squares across and down and multiplied." He then wrote and partially erased several incorrect equations in various quadrants in Task 2 (see Figure 13).

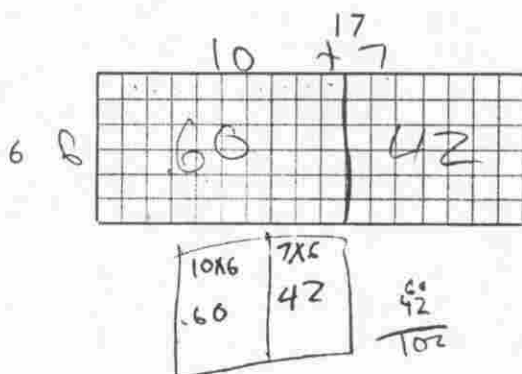


FIGURE 12 Pete's solution to 6×17 .

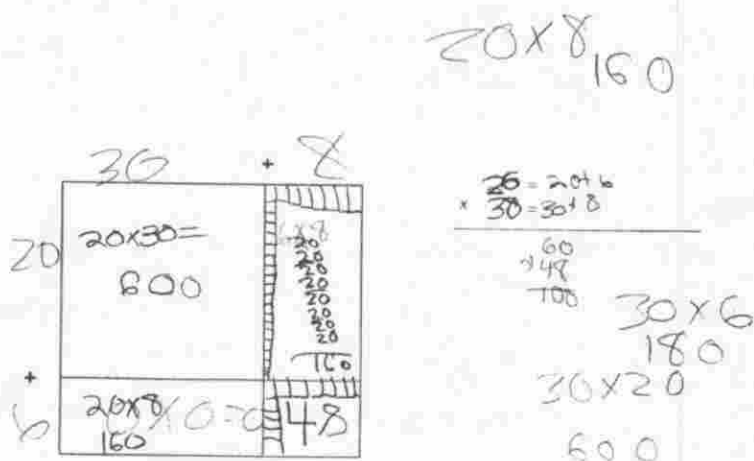


FIGURE 13 Pete determined areas. Note that to make Pete's final work in the bottom-right-hand quadrant readable, I have deleted " $0 \times 0 = 0$," " 30×6 ," and "180," initial inscriptions that he erased.

Pete connected quadrants and unit squares representations. When I asked Pete once more about his solution to 6×17 , he explained that 60 and 42 told him how many unit squares were in each region. I reminded him that numbers in quadrants were also supposed to reflect "the number of unit squares." Pete agreed and immediately explained connections between unit squares and quadrants representations:

A1 Pete: The answer is the amount of unit squares in here (pointed to the unit squares representation for Task 1), right? So, this is (pointed to the quadrants representation for Task 2), see this is just 1 big, like, thingy of this (pointed to the unit squares representation). The easiest way for me to figure out like a problem like this (pointed to the unit squares representation) is just make this (pointed to the unit squares representation), like this (pointed to the quadrants representation), like all these little square units in there (pointed to the quadrants representation), just make it look like this (pointed to the unit squares representation) and I can figure it out.

A2 Int: Umm, I didn't quite follow that, I got a little conf.... I got a little confused. Can you repeat it please?

A3 Pete: Imagine this (pointed to the unit squares representation),

A4 Int: Mm.

A5 Pete: like this thingy (pointed to the quadrants representation), like being a box with square units inside.

Analysis. Pete could explain some connections between the quadrants and unit squares representations, but needed new strategies for determining areas when unit squares were no longer present. Although Pete could not write appropriate equations for quadrants in Task 2, his at least partial connections between unit squares and quadrants representations were particularly clear when he said that the quadrants representation in Task 2 was just a "big" version of the unit squares representation (line A1), and when he asked me to imagine the quadrants representation with "square units inside" (line A5).

Pete determined areas of quadrants after drawing some unit squares. Pete identified what made quadrants representations hard for him to use, "There's no square units inside so it's hard to, um, multiply 'cause I usually count." He then introduced the problem 17×18 and counted imaginary unit squares along edges of the 10-by-10 region in the quadrants representation to determine "that's 100." Pete explained that he would "do the same" with the remaining three quadrants and "add 'em up and get the answer."

I asked Pete to draw unit squares in the bottom right-hand corner of the quadrants representation for 26×38 . He pointed to the 8 and 6 in his numeric labels, said that there would be 8 squares across and 6 down, and drew these in (see Figure 13). When I asked how many unit squares were in that quadrant, Pete answered "48" immediately and wrote this number in the quadrant. I asked how he found this out, and Pete explained that "you multiply the top (traced the row of unit squares he had just drawn) and this (traced the column of unit squares he had just drawn)."

Next I asked Pete to make another quadrant "easier to work with," and he drew 8 and 21 unit squares along the edges of the 20-by-8 region. (I did not notice that Pete had drawn an extra square.) When I asked why these were the right numbers of unit squares, Pete pointed to the 8 and 20 in his numeric labels. Then, to calculate the total number of unit squares, he first added eight 20s and then multiplied 20×8 . Instead of drawing unit squares for the final two quadrants, Pete calculated 30×20 and 30×6 to the side of the quadrants representation. Pete did not change his 20×8 calculation for the bottom left-hand quadrant and appeared confused when I focused his attention on the discrepant answers. Pete did not have time to find the final product.

Analysis. Data in this section suggested that Pete evolved old strategies that relied on unit squares into new ones that relied on numeric labels. Pete may not have attended initially to the connection between numeric labels and numbers of unit squares along edges of quadrants, but he gestured immediately toward the 6 and 8 in his numeric labels when I asked him to draw. Pete used his drawn unit squares to determine the area of the 6-by-8 region in the same way that he used unit squares in his solution to 6×17 . How Pete used drawn unit squares to determine the area of the 20-by-8 region was less clear because of the extra unit square, but he clearly calculated 160 in two different ways and so perhaps miscounted. I supported Pete's accomplishment by asking him to recall the unit squares strategy that he understood and to modify the quadrants representation so that he could apply that strategy once more. The numeric label strategy that Pete arrived at may have emerged by condensing a two-step strategy in which he used labels to determine how many unit squares to draw, and then used drawn squares to determine areas.

Although Pete apparently used new representational features to determine areas, his confusion about discrepant answers for the bottom left-hand quadrant cast doubt on the stability of his strategy. Pete's initial difficulties suggested that he needed more assistance understanding connections between numeric labels and unit squares like those that Mrs. Tate had discussed for the 8-by-4 region during the class solution to 28×34 and for the 20-by-60 region during the class solution to 25×63 .

Jodi's Work on Task 2

Jodi had difficulty using labels to determine areas of quadrants. Jodi read the drug store problem, wrote " $10 + 10 + 10 + 8$ " across the top of the quadrants representation, and wrote " $10 + 10 + 6$ " down the left-hand side. Figure 14

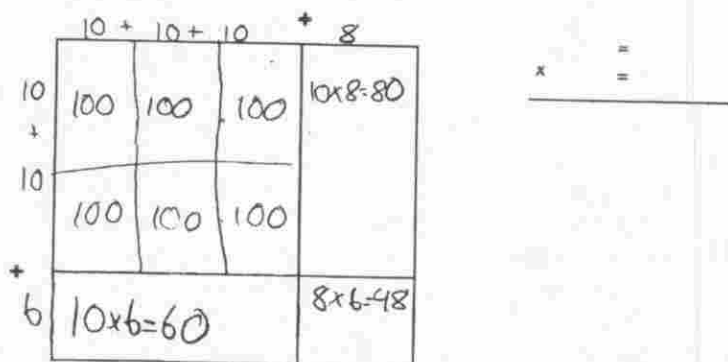


FIGURE 14 Jodi labeled the quadrants representation, attempted areas, and completed correct numeric methods.

shows that she drew horizontal and vertical lines across the top left-hand quadrant, wrote "100" in each of the resulting 100 squares, wrote " $10 \times 8 = 80$ " in the top right-hand quadrant, wrote " $10 \times 6 = 60$ " in the bottom left-hand quadrant, and wrote " $8 \times 6 = 48$ " in the bottom right-hand quadrant. She also completed the expanded algorithm and lattice methods correctly.¹⁰

Note Jodi's use of 10s in numeric labels as she explained her work:

B1 Jodi: I did 10 times 10 (pointed to the first 10 in $10 + 10 + 6$ and the first 10 in $10 + 10 + 10 + 8$), which is 100 (pointed to the top left 100 square). And then I kept on doing that (pointed back and forth between $10 + 10 + 6$ and $10 + 10 + 10 + 8$). And all of these were 100 (pointed to the remaining 100 squares).

B2 Int: OK.

B3 Jodi: And then what I did is I did 8 times 6 (pointed to the 6 in $10 + 10 + 6$ and the 8 in $10 + 10 + 10 + 8$), which went here (pointed to the bottom right-hand quadrant), and then 10 times 6 (pointed to the third 10 in $10 + 10 + 10 + 8$ and the 6 in $10 + 10 + 6$), which went here (pointed to the bottom left-hand quadrant). And then 10 times 8 (pointed to the second 10 in $10 + 10 + 6$ and the 8 in $10 + 10 + 10 + 8$), which went here (pointed to the top right-hand quadrant).

Jodi finished her comments by explaining that she used two numeric methods, each of which gave the same answer. Her comments did not suggest that she compared her numeric methods against her work with the quadrants representation.

After Jodi's partner, Alice, explained how she determined areas for various regions in her work, I began to investigate why Jodi did not determine correct areas for the 20-by-8 and 6-by-30 quadrants by asking how many unit squares were "supposed" to go across the top and down the side of several regions. Jodi said that there would be 10 squares across and down for 100 squares, and 30 squares across and 6 down for the 6-by-30 quadrant. She continued pointing to the third 10 in $10 + 10 + 10 + 8$ as she gave further explanations for her $10 \times 6 = 60$ equation.

C1 Jodi: Well what I, what I did is since, I just, I, (pointed to the 6 in $10 + 10 + 6$) if I do, since there were 10s up here (pointed toward the $10 + 10 + 10 + 8$ label), and how I got 10 times 6, was (pointed to

¹⁰The lattice, or Galosia, method was introduced one day when the school principal demonstrated an *Everyday Mathematics* lesson (University of Chicago School Mathematics Project, 1995).

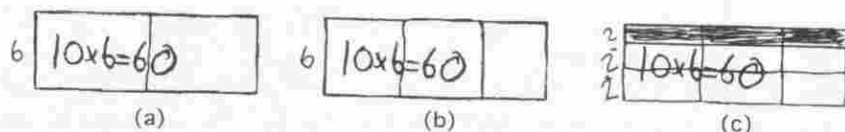


FIGURE 15 Jodi divided the 6-by-30 region three ways.

the third 10 in $10 + 10 + 10 + 8$) I, like I just ... (pointed to the 6 in $10 + 10 + 6$) 10, 6 times, well 10 times 6 (pointed to the third 10 in $10 + 10 + 10 + 8$ and the 6 in $10 + 10 + 6$) because if it was, I wouldn't do it 6 times 8 (pointed to the 6 in $10 + 10 + 6$ and the 8 in $10 + 10 + 10 + 8$) because that would go there (pointed to the bottom right-hand quadrant).

C2 Int: Mm.

C3 Jodi: But if I did, and 10 times (pointed to the third 10 in $10 + 10 + 10 + 8$), and if there, there's 10 going across in each of the 100 boxes (pointed to each of the three 100 squares directly over the 6-by-30 quadrant), so I thought about doing 10 times 6 (pointed to the 6 in $10 + 10 + 6$ and the third 10 in $10 + 10 + 10 + 8$), instead of splitting it up into like 10, 10 times like 3, because then it will be, like, it will be 90, if I did 10 times 3.¹¹ But if I, if I, if I made this smaller, if (split the 6-by-30 quadrant with a vertical line, see Figure 15a), if I cut this in half, it'll be 10 times 3 (pointed to the second 10 in $10 + 10 + 10 + 8$ and the 6 in $10 + 10 + 6$), and then 10 times 3 (pointed to the third 10 in $10 + 10 + 10 + 8$ and the 6 in $10 + 10 + 6$), but since I'm doing 26 (pointed to the 6 in $10 + 10 + 6$), it was just one 6 (erases the line that she just drew).

After this explanation, I asked Jodi again how many unit squares were "supposed to fit" along the top edge of the 6-by-30 quadrant. Jodi said "30." I then asked her if the "equation" was 10×6 , and Jodi said, "Yeah."

Analysis. Jodi began Task 2 by adding numerical labels and lines like those in Figure 8 and in the second set of class solutions for finding groups ("Broke factors into single 10s and 1s directly" and "Drew lines from plus signs" in Table 1). Having introduced the $10 + 10 + 10 + 8$ and $10 + 10 + 6$ labels, and having drawn

¹¹I could not tell how Jodi arrived at 90, but she appeared to drop this as she continued her explanation.

lines across just the 20-by-30 quadrant, Jodi had then to coordinate numeric labels with areas of sub-regions in complex ways. She did not explain why she subdivided just the 20-by-30 quadrant, but may have recalled an atypical class solution from 1 week before this interview. During the solution to 66×48 , Mrs. Tate drew lines across just the 60-by-40 quadrant because the instructional materials included a 100s/10s/1s representation in which all other lines for groups of 10s and 1s were printed.

One explanation for why Jodi arrived at 10×6 repeatedly (lines B3, C1, and C3) was that she did not understand the dimensions of the 6-by-30 quadrant, but she twice stated that there were 30 unit squares across the top and once stated that there were 6 down the left-hand side. A second explanation was that Jodi momentarily forgot the dimensions, but she explained that there were 10 unit squares across each of three 100 squares and then restated her incorrect 10×6 expression (line C3). These data strongly suggested that Jodi understood the number of unit squares along the edges of the 6-by-30 quadrant, yet could not determine the area.

A third explanation for why Jodi arrived at 10×6 was that she did not fully understand how to use her numeric labels. Jodi consistently used individual addends in her $10 + 10 + 10 + 8$ and $10 + 10 + 6$ labels. This use of labels was correct for the top left 100 square and the 6-by-8 quadrant (lines B1 and B3), but she extended this use inappropriately to the 6-by-30 and 20-by-8 quadrants. She consistently gestured toward the third 10 in $10 + 10 + 10 + 8$ when discussing $10 \times 6 = 60$ (lines B3, C1, and C3) and gestured once toward the second 10 in $10 + 10 + 6$ when discussing $10 \times 8 = 80$ (line B3). Moreover, after dividing the 6-by-30 quadrant in two, Jodi pointed to different 10s in $10 + 10 + 10 + 8$ and may have used separate, imagined 3s when calculating 10×3 for each sub-region (line C3). Jodi did not appear to notice the discrepancy between her equations for quadrants and the 30×6 and 20×8 calculations in her numeric methods. She could use numeric methods correctly, but did not connect them fully to the area context.

Jodi's difficulties continued. During the balance of the interview, Jodi introduced a second and third strategy for determining the area of the 6-by-30 quadrant. Neither was correct. For her second strategy, she extended her drawn vertical lines from the 20-by-30 quadrant, dividing the 6-by-30 quadrant into three parts (see Figure 15b). She explained that she would calculate 10×2 for each sub-region, because the resulting 20s would still add to 60. I asked for the number of unit squares along the edges of one of these regions, and Jodi answered 6 and 10. When I asked how many unit squares would be inside, Jodi appeared to experience conflict and answered, "Oh. 60."

Jodi then introduced her third strategy by erasing the 6 in her $10 + 10 + 6$ label, writing " $2 + 2 + 2$ " in its place, and drawing two horizontal lines across the 6-by-30 quadrant (see Figure 15c). She said, "Now it'll be 10 times 2." I asked Jodi to shade

the region to which she referred. She shaded the topmost 2-by-30 region as shown in Figure 15c, but explained that "since there's 10 going across (traced the top edge of the leftmost shaded 2-by-10 region) and 2 going down (traced the left-hand edge of the same region), it'll be 10 times 2." When I asked Jodi if 10×2 was for the whole shaded region, she said no "because there's three 10s going across (pointed to each 10 in her $10 + 10 + 10 + 8$ label), which is 30." Jodi changed her computation for the shaded region to 30×2 , added 60 three times, and arrived at 180.

Jodi was not convinced that her new approach was correct, however, and for the balance of the interview favored her original $10 \times 6 = 60$ equation. When Alice explained how she used $10 \times 6 = 60$ for each of three regions that resembled those in Figure 15b, Jodi argued, "I think it should only be one [60] because it's, it's, you're only timesing one, you are only like timesing one 6 (pointed to the 6 in Alice's label)." When I asked Jodi why, she responded, "Because there's only one 6 (pointed to the 6 in Alice's label)."

Analysis. Jodi struggled to coordinate three understandings about the 6-by-30 quadrant. The first correct understanding was about numbers of unit squares along edges of regions. In an earlier section, Jodi stated that there were 30 unit squares across and 6 down. In an earlier section, she stated that there were 10 unit squares across the top and 6 down the side of one 6-by-10 region in Figure 15b, and that there were 10 unit squares across the top and 2 down the side of one 2-by-10 region in Figure 15c. The second understanding was about using numeric labels to determine equations. Jodi continued to use individual addends from the horizontal and vertical labels when determining equations. When determining the equations for the three subregions in the 6-by-30 quadrant (Figure 15b), Jodi broke apart the 6 in $10 + 10 + 6$ into three 2s and appeared to use a different 2 and 10 for each 2-by-10 region. Moreover, when questioning Alice's approach, Jodi explained that you should only multiply one 6 because there was only one 6 in Alice's label. She did not seem to notice that she had initially used 6 both in her $10 \times 6 = 60$ and in her $8 \times 6 = 48$ calculations. The third understanding, made clear by Jodi's methods shown in Figure 15a and 15b, was that the "answer" for the 6-by-30 quadrant should be 60. This understanding was incorrect and obstructed her coordination of unit squares and numeric labels. Had she focused more on her calculations for the expanded algorithm and lattice methods, she might have overcome this difficulty. Jodi remained focused on determining areas and did not discuss finding final products during her interview.

Maria and Tom

Maria and Tom also had difficulty determining areas of quadrants. Like Pete, Maria relied on counting unit squares to determine areas and so could not use the quadrants representation. Tom reversed the products for the 6-by-30 and

20-by-8 quadrants. He explained that he used large factors (i.e., 20 and 30) for the large quadrant and small factors (i.e., 8 and 6) for the small quadrant.¹² Because Tom relied on correspondences between relative sizes of factors and quadrants, he may not have seen any difference between putting a given medium sized product in one medium sized quadrant or the other. His strategy led to correct answers, although it did not reflect correct connections between numeric labels and areas of quadrants. Tom may have based his strategy on an explanation that Mrs. Tate provided during one whole-class solution a week prior to his interview. In that explanation, Mrs. Tate told students to use the numbers from the biggest region in the 100s/10s/1s representation (i.e., tens \times tens) for the biggest quadrant in the quadrants representation.

The Final Four Students

The remaining 4 students I interviewed could not complete Task 1, because they struggled with connections among numeric labels, unit squares, and areas of regions. Two of these students received services for learning disabilities in mathematics. Some could not add appropriate labels to the unit squares representation in Task 1. Those who could often had to count all unit squares in a given region to determine the area.

Summary

Pete, Jodi, Maria, and Tom's work on Task 1 made clear that these students could make appropriate connections among numeric labels, unit squares, and areas of rectangles when unit squares were present. These same students' work on Task 2 suggested that making connections within and among representations, though necessary, can still be insufficient for solving problems. In particular, these students struggled to coordinate representational features with the goal of determining areas of quadrants. Pete struggled to find dimensions when unit squares were not present, but began using strategies that relied on numeric labels during his interview. Jodi and Tom added appropriate numeric labels for dimensions, but struggled to coordinate addends in those labels with areas. Jodi's challenge appeared to be coordinating the groups of imaginary unit squares that she formed by drawing vertical and horizontal lines and her use of individual addends in numeric labels. The performance of the final four students made clear that more time and support may be necessary if all students are to find accessible an approach to two-digit multiplication based on areas of rectangles.

¹²This is the explanation that may have led Pete to erase his $0 \times 0 = 0$ equations.

STUDENTS' PERFORMANCE ON THE END-OF-UNIT TEST

The end-of-unit test complemented the interview data by providing data on students' computation performance. The test asked students to solve the problems 17×12 , 45×26 , 37×24 , and 92×78 and "to draw any pictures that you need." The percentage of Mrs. Tate's students who answered each problem correctly were 88%, 80%, 80%, and 64%, respectively. These are strong results compared to the 54% of U.S. fifth-grade students who solved 45×26 correctly in the Stigler et al. (1990) study cited at the outset of this article. Mrs. Tate's students answered the four items using one or more of the following three methods: the expanded algorithm, the traditional U.S. method, or the lattice (Gelosia) method. Again, the second method was brought into Mrs. Tate's class by students, and the third was introduced by the school principal. The distribution of methods that students used was similar for each of the four items. Although not required to do so, five students drew quadrants representations for at least one problem and, in all but one case, arrived at correct answers.

For the item 45×26 , 19 of Mrs. Tate's 25 students (76%) used the expanded algorithm. Of these, 10 (40%) used just the expanded algorithm and used it correctly, 4 (16%) used both the expanded algorithm and the traditional U.S. method correctly, and 1 (4%) used the expanded algorithm and the lattice method correctly. The remaining 4 (16%) students who used the expanded algorithm set up correct factors for all four partial products. One student computed all partial products correctly except $5 \times 6 = 48$, one student computed all partial products correctly except $40 \times 20 = 8000$, one student computed all partial products correctly except $40 \times 20 = 8000$ and $40 \times 6 = 100$, and one student got the correct answer but reversed partial products for quadrants in the same way that Tom did. Mrs. Tate's remaining 6 (24%) students used the traditional method. Of these, 4 (16%) used just the traditional method and used it correctly, 1 (4%) used the traditional and lattice methods correctly, and 1 (4%) began the traditional method, determined 6×45 , but was unable to compute 20×45 . These results demonstrate that by end of the CMW two-digit multiplication unit a large majority of students could compute correctly with the expanded algorithm, and many could use a second method as well. Note that for some students, including Jodi, the ability to perform correct computations may have preceded the ability to fully connect numeric methods to areas of rectangles.

DISCUSSION AND IMPLICATIONS

Analysis of whole-class solutions to two-digit multiplication problems revealed that Mrs. Tate and her students contributed to a range of class strategies, particularly for determining areas in unit squares and 100s/10s/1s representations (Re-

sult 1), and that these strategies eventually converged on those that coordinated magnitudes of partial products, expanded forms for factors, and the distributive property (Result 2). The analysis pointed to challenging decisions that teachers must make about the range of strategies they pursue with their students. Some teachers may value having all students use the same approach to solve a class of problems, such as two-digit multiplication problems. Although emphasizing one approach might minimize some sources of student confusion, there are important reasons for exploring multiple approaches. A range of approaches can make topics or problems accessible to students with diverse understandings, can provide a context in which to discuss conditions under which each approach might be advantageous, and can afford opportunities to make connections among different representations of the problem context. Thus, multiple approaches can make mathematical connections and flexible problem solving realizable in heterogeneous classrooms.

The analysis of Mrs. Tate's lessons not only illuminated such opportunities but also raised questions about when teachers should direct students toward particular strategies. If Mrs. Tate provided too little guidance, then her students might not coordinate expanded forms for factors and repeated groups, and hence might not develop understandings of efficient and general multiplication methods. If she moved the class to particular groups too quickly, however, then students might have a hard time participating if the method they understood dropped from class discussions. Despite the challenges of pursuing multiple problem-solving strategies in classrooms, affording a range of initial approaches and managing convergence toward target strategies may be a useful design principle for instructional materials that aim to support conceptual and procedural understandings of core topics.

Analysis of end-of-unit interviews revealed the different degrees to which students had connected numeric methods to areas of rectangles (Results 3 and 4) and further learning issues that were not evidenced in whole-class discussions (Result 4). These results complement those of Outhred and Mitchelmore (2000) by suggesting that many fourth-grade students can use multiple representations and the array structure of equal rows and columns to develop conceptual understandings of multidigit multiplication. Whereas Outhred and Mitchelmore found that fourth-grade students could cover rectangles by drawing arrays of unit squares, suggesting students' intuitive understanding of area measurements when dimensions are whole numbers, this study found that some fourth-grade students could connect dimensions of rectangular arrays to numeric labels more readily than others. In particular, even though Pete and Jodi were present for the class solutions to 28×34 and 25×63 , during which Mrs. Tate discussed connections between strategies that relied on unit squares and those that relied on numeric labels, these students' difficulties revealed that some students did not readily make similar connections. Thus important aspects of teaching and learning with the CMW two-digit

multiplication materials did not surface in whole-class discussions. More explicit discussions identifying particular representational features, connections among such features, and ways that different features can be used to accomplish similar goals may have helped more of Mrs. Tate's students connect the geometric and numeric representations. Such results demonstrate the detail with which analyses need to be conducted if they are to provide insight into processes of teaching and learning with multiple representations.

Results of this study suggest that revised CMW two-digit multiplication materials could benefit more students. The repeated groups on which some students focused, particularly in the 100s/10s/1s representation, afforded coordination of magnitudes of factors and products, but not expanded forms for factors and the distributive property. The sequence of problems in the CMW materials has been revised and restrict 100s/10s/1s representations to a few lessons that focus on products like $20 \times 30 = 600$ and $8 \times 30 = 240$. The intent is for students to connect dimensions of six 100 squares or three 80 rectangles with areas of those regions, to understand the advantages of using repeated groups that coordinate expanded forms for factors and hence emphasize place value, and to use those understandings as the basis for solving more complex multiplication problems.

Finally, as mentioned at the close of the previous background section, Mrs. Tate did not focus on the same norms for justifying solutions as did teachers at the center of previous studies that have used the emergent perspective. Thus, when Mrs. Tate's students stopped asking questions about strategies for finding groups and determining areas, those strategies became beyond justification and thus met the standard for taken-as-shared in this classroom. Nevertheless, the end-of-unit interviews revealed that Mrs. Tate's students understood class strategies to very different degrees. Had norms been established in Mrs. Tate's classroom that afforded more opportunities for students to explain strategies in full, and that required students to take greater responsibility for asking questions when they did not understand, then the standard for taken-as-shared might have been higher and taken more time to achieve. This observation suggests the hypotheses that norms and classroom mathematical practices are tightly connected, that different configurations of norms can lead to different means by which practices emerge, and that the means by which practices emerge can shape in fundamental ways relationships between individual learning and that of the classroom community. Such hypotheses should be the subject of future research.

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