PATHWAYS TO NUMBER
Children's Developing Numerical Abilities

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6 Relationships Between Counting and Cardinality From Age 2 to Age 8

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Young children begin to understand and to use number words in seven different kinds of contexts (see Fig. 6.1). Three of these contexts are mathematical ones: a cardinal context, in which the number word refers to a whole set of entities (a discrete quantity) and describes the manyness of the set ("I want two cookies"); an ordinal context, in which the number word refers to one entity within an ordered set of entities and describes the relative position of that entity ("I was second"); and a measure context, in which the number word refers to a continuous quantity and describes the manyness of the units that cover (or fill) the quantity ("I am two years old," perhaps with two fingers showing, making it also a cardinal context). Two other contexts, sequence and counting, provide cultural tools for ascertaining the correct number word to be used in cardinal, ordinal, or measure contexts. The sequence context is a recitation context in which number words are said in their correct order but no entities are present, and the number words refer to nothing; this context is originally like reciting the alphabet or the days of the week. In the counting context, number words are put into a one-to-one correspondence with entities; each number word refers to a single entity but describes nothing about it (it is just a count label, or tag, for the entity). Number words are also used to say written numerals. This symbolic context (or perhaps better, a numeral context) originally elicits a number word with no accompanying meaning and no reference beyond the numeral itself ("That's a six" upon seeing *6*). Later on, written numerals themselves can take on cardinal, ordinal, measure, counting, or sequence meanings. Finally, number words are also used in non-numerical (or at least quasi-numerical) contexts, such as telephone numbers, television channels, zip codes, house addresses, and bus numbers.

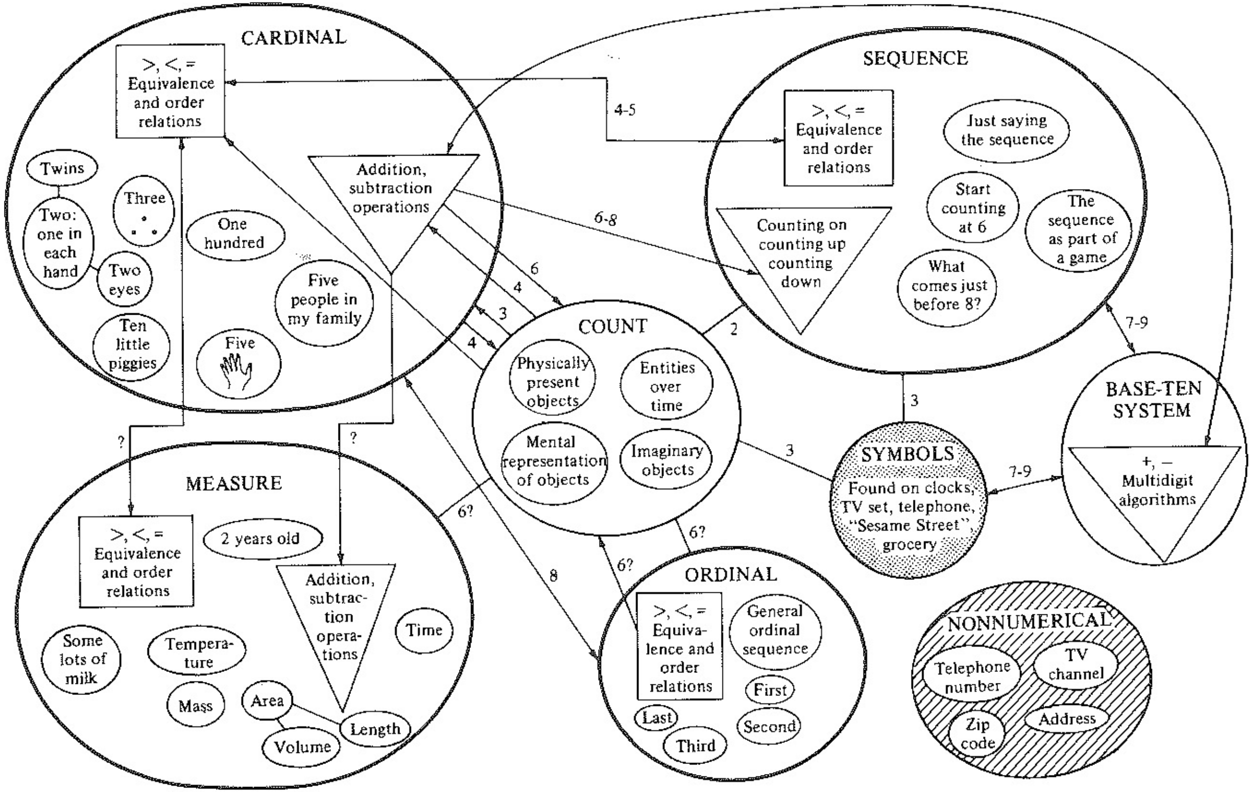


FIG. 6.1. Relationship among cardinal, ordinal, measure, sequence, count, symbol, and nonnumerical contexts (numbers on connecting lines are approximate ages).

Young children hear number words being used in these seven different contexts and begin to use number words themselves in these various different contexts. These meanings are originally separate meanings for children. Gradually a child begins to make connections among these various meanings and a single spoken number word then may take on more than one meaning simultaneously. Learning all of these relationships takes a long time, from age 2 to about age 8 for most children. This chapter summarizes the developmental relationships that children construct among these various meanings. This construction culminates, finally, in a seriated, embedded, unitized, cardinalized sequence of number words, a postconservation construction related to what Piaget called "truly numerical counting." The evidence supporting the developmental paths to be described here is discussed in my book, *Children's Counting and Concepts of Number* (Fuson, 1988), where work of other researchers working with children from the United States and England is also discussed; parts of this chapter reflect thinking that has progressed since the book was completed. Major elements of the construction of relationships among all of these different meanings of number words seem to be shared by children in most cultures. For example, the main developmental path to the understanding of addition and subtraction followed by most children in the United States is shared by children in the Soviet Union (Davydov & Andronov, 1981) and Oksapmin children in New Guinea (Saxe, 1982). There also seems to be another related path taken by some children in Sweden and by children in Asian countries; this alternative path and differences between that path and the path described here are discussed by Fuson and Kwon (this volume, chap. 15). The path to be described here is not supported by teachers in classrooms in the United States—in fact, many teachers have traditionally tried to suppress this path while offering nothing but memorizing facts to replace it—but the evidence is quite robust that many children in the United States independently construct this path for themselves as a way to give meaning to numerical situations.

This book is a celebration of Piaget's book on children's construction of number. This chapter and the work it summarizes are framed within that Piagetian work on number. It assumes that each child must construct his or her own path through increasingly complex number concepts, and the Piagetian stages of conservation of numerical equivalence describe some of these increasing complexities. My own work has concentrated on understanding how children come to understand numerical situations with numbers that are too large to process perceptually, that is, numbers greater than six. This has led to a concentration on the cultural tool of counting because counting is used by children in constructing cardinal, ordinal, and measure number concepts for all but very small and very large sets. Thus, much of my work has focused on contexts of specified numerosities rather than the contexts of unspecified numerosities studied by Piaget. My work, and the related work of others, has attempted to fill gaps in the Piagetian logical account of the construction of number. It has pointed out various critical roles that counting plays in children's developing understanding of number, and shown that the Piagetian framework underestimated the importance of these roles of counting. Piaget's account also underestimated the role that the empirical strategy of matching can play, in spite of Piaget's emphasis on one-to-one correspondence, but relatively little research has been done on matching as opposed to counting. However, my work and that of others does support the general Piagetian position that counting alone is not sufficient for an adequate understanding of number and that, in transformed situations, operational thinking moves beyond counting and matching. Thus, we now have a richer and more complete view of the complementary roles of the cultural tools of counting and matching and of general operational, quantitative thinking.

In order to understand young children's thinking about numerical situations, it is important to clarify some common errors in the use of the word *ordinal.* This word is used to refer to three different meanings of number words: sequence meanings, count meanings, and ordinal number meanings. An *ordinal number* refers to a context in which the entities are ordered (such as in a queue), and the number refers to the relative position of that entity. Many languages signify this special numerical context by using entirely different number words or by adding special letters to the usual counting words (in English the counting words are *one, two, three, four, five, . . . ,* whereas the ordinal words are *first, second, third, fourth, fifth, . . . ,* with most later ordinal words made from counting words by adding *th*). The very fact that two different lists of words are used clearly indicates that the culture differentiates between counting contexts (in which counting words are used) and ordinal contexts (in which ordinal words are used). Another difference between count and ordinal contexts is that an ordinal context has one given, unchangeable order. In counting contexts, one must impose an order on the entities to count them, but one can make many different orders on those entities and count a set in many different ways. Thus, in a counting context, an entity can take any given count word, whereas in an ordinal context a given entity can take only its single correct ordinal word, according to the given ordering. A *sequence context*—saying the number words in their standard order—is also sometimes called an ordinal context because number words in every language have a single correct order. This order does create sequence meanings for number words, and certain relationships derive from this single correct order, but these sequence meanings are like the meanings that accrue to any ordered list—the alphabet or the months of the year—and are not originally quantitative. The word *ordinal* is also used erroneously to refer to order relations (greater than, less than) on cardinal numbers (seven is more than four). Order relations ($>$, $<$) can be established on cardinal, ordinal, or measure numbers or number situations and on sequence words (seven comes after four). It is considerably easier to understand and describe the task of the child in constructing a developmental path of numerical concepts if these three meanings—ordinal, count, and sequence—are differentiated clearly and used accurately and if order relations are labelled as such and the type of number word meaning used in the order relation is specified.

There is relatively little work on measure and ordinal number contexts. Walter Secada and I did some unreported work on conservation of ordinal number in which toy animals were lined up in a queue to go into cages and one queue was then transformed to be longer or shorter than the other. We found that children who were at Stage 2 of conservation of cardinal number (the traditional Piagetian conservation of numerical equivalence task) had great difficulty with this ordinal task, partly because they did not know the ordinal words, but many also had difficulty when ordinal words were not used. In children in the United States knowledge of the ordinal words lags behind knowledge of the counting words by years (Beilin, 1975), so much ordinal number knowledge seems to lag behind cardinal knowledge. Understanding of measure contexts also lags considerably behind knowledge of cardinal contexts because measure contexts require understanding the unit of measure (and the inverse relationship between the size of the unit and the measure number of the quantity), whereas much understanding of cardinal contexts can be accomplished by a child who uses only perceptual unit items in which each entity in the cardinal context is taken as an equal, single entity (see Steffe et al., 1983). Because most cardinal relationships are constructed before ordinal and measure relationships are constructed, I concentrate in this chapter on the relationships constructed by children among sequence, counting, and cardinal meanings of number words.

EARLY CONSTRUCTIONS: PATTERN NUMBER WORDS AND COUNTED NUMBER WORDS

Young children's first cardinal uses of number words refer to small numbers of entities and seem to rest on *subitizing*, the immediate apprehension of small numerosities. There is controversy concerning the basis for subitizing; the ages at which children can subitize two, three, and four entities; and the developmental relationship between subitizing and counting. What is clear is that young children do learn to subitize at least two entities and that many children learn to label particular patterns or situations with a cardinal label (e.g., "There are five people in my family."). Children continue to use this pattern-based approach of seeing certain situations as patterned sums of small numbers of entities (e.g., After I cut a peanut butter sandwich in half, and in half again, to make four small squares, my daughter aged 2 years, 10 months said, "Two and two make four."). These special pattern-based small numerosities continue for several years to play important roles in some equivalence, addition, and subtraction situations.

Before children can count entities, they must learn the correct sequence of number words. Errors made in learning this sequence seem to depend on the structure of the sequence. In English the irregularities in the number words after *ten—eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty*—seem to hide even the irregular relationship of the *-teen* words to the words before *ten;* thus, most children learn the sequence of words to *twenty* as a rote list of meaningless words, much like the alphabet. The incorrect sequences produced by 3- and 4-year-olds in the United States possess a typical structure (see Fuson et al., 1982): They consist of a first portion of number words in their correct order, followed by a stable portion that is not correct, followed by an unstable portion that varies each time number words are said. The stable incorrect portion usually consists of words in the correct sequence but with some omitted (e.g., "11, 12, 13, 16" or "13, 14, 16, 18"); reversals of word order are uncommon. These incorrect stable portions may be said by a given child for several months or even longer.

Most middle-class children in the United States below the age of 3½ are just learning the sequence to 10, those between 3½ and 4½ have correct sequences to 10 but have incorrect portions somewhere between ten and twenty, and many between 4½ and 6 are working on the decade structure between twenty and one hundred, although a substantial proportion of children in this age range may still have incorrect portions in the upper teens (see Fuson, Richards, & Briars, 1982). As soon as children's accurate portions reach into the 20s, their sequences show evidence of understanding of the decade structure of the English words (the *x-ty, x-ty–one, x-ty–two, . . . , x-ty–nine* pattern), but it takes them a very long time to learn the decade words themselves (*twenty, thirty, forty, . . . , ninety*) in their correct order. They produce a series of *x-ty* to *x-ty–nine* chunks that may be out of order, and may even repeat chunks for as long as a year and a half.

Two-year-olds often begin to count objects. They typically point to objects and say number words. Counting entities distributed in space (as opposed to counting entities occurring over time, such as clock chimes) requires an indicating act, such as pointing, to connect the words said over time to the entities distributed in space. Pointing (and other indicating acts, such as moving objects into a counted pile or eye fixation on particular entities) isolates a particular spatial location at a particular moment of time. It thus creates spatial–time units that enable a one-to-one correspondence in time to be made between the points and the spoken number words, and a one-to-one correspondence in space to be made between the point locations and the entities. If each of these correspondences is correct, the counting is accurate. Preschool children make errors that violate each of these correspondences in both possible ways: They make a point without saying a word and say a word without making a point; they give extra points to an object and leave some objects without any points. They also occasionally produce complex combinations that violate both correspondences (e.g., give an object three words and two points) and produce degenerative pointing (a skimming across the objects with the finger while saying words at random).

The rates at which 3-, 4-, and 5-year-olds make these errors, and how object characteristics of number, color, homogeneity, and arrangement affect error rates is discussed in several chapters in Fuson (1988). Preschoolers show surprising competence in creating correct correspondences in counting objects arranged in rows, with children aged 3 to 3½, 3½ to 4, and 4 to 4½ making correct correspondences on 84%, 94%, and 97% of the objects, respectively, in rows of 4 to 14 objects. The error rate increases with longer rows, falling to 56%, 64%, and 71% of the objects correct, respectively, in rows up to 32 objects. Counting accuracy also varies considerably with how hard the child is trying to count accurately (i.e., with effort). When objects are maximally disorganized, children not only have to carry out the local time–space number word–point–object correspondences, but must also create a global correspondence over all the objects, so that each object is counted and no object is re-counted. This requires the child to use either a remembering strategy, to keep track of which objects have been counted, or a physical strategy, such as moving objects into an already counted pile, so that remembering is not necessary. Young children are much less successful at creating such a global correspondence than they are at carrying out the local correspondences in a linear set of objects, and 5-year-olds still make many re-count or uncounted-object errors on large, disorganized arrangements having 10 to 30 objects.

When young children first begin counting, the counting does not have a cardinal result. They count only to imitate the social-cultural counting activity. If asked how many there are after counting, children re-count (and continue to re-count each time they are asked "how many?"), they say a number word that is not their last counting word, or they say a sequence of number words (often different from the words they said in counting). It is not clear how children first learn the relationship between counting and cardinality and about how much understanding of cardinality is indicated even when children answer a "how many?" question with the last word they said in counting. My own work (summarized in Fuson, 1988, chap. 7) indicates that different children may use different ways to make the connection between their last counted word and that word as indicating how many objects there are. In most cases children then generalize this relationship fairly rapidly and use this relationship across sets of different sizes. Some children may count a very small set and also subitize that set, and then notice that the last word said in counting is the same as the subitized numerosity. Something may make the last counted word particularly salient to the child, and that child may then answer a "how many?" question with that word. Children may be told by a parent or sibling that their last counted word tells how many objects there are, and this may be sufficient for them to start responding correctly. Young children's recency bias (their tendency to answer a multiple-choice question with the last listed answer) may also contribute to their answering a "how many?" question with the last word they say. Failing to remember their last counted word does not seem to be a major reason children fail to answer a "how many?" question with their last counted word: Most 2- and 3-year-old children who do not answer the "how many?" question correctly *do* remember the word they said for the last object in a row.

Many children who do answer a "how many?" question with the last counted word seem to have constructed only a last-word rule, in which that last word does not refer to the whole set and does not refer to the numerosity of that set. Children give the last counted word even when their counting is very inaccurate and yields a last-word response that is considerably discrepant from the cardinal meaning of that response: for example, counting a set of 26 objects: "1, 2, 3, 6, 7, 8, 9, 1, 2." "How many are there?" "Two," or counting a set of two objects: "1, 2, 3. There are 3."; or repeating a given number several times in counting, but still giving it as the last-word response: counting a set of 23: "1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 14, 15, 16, 14, 16, 15, 15, 16, 11-teen, 15." "How many are there?" "15." Many 3-year-olds giving last-word responses nevertheless show confusion between counting and cardinality meanings, and they use cardinal plural forms to refer to the last object, that is, as a count reference, rather than as a cardinal reference to all the objects. Three examples reported in Fuson (1988) are "Those are the five soldiers" as the child points to the last soldier; "This one's the five chips" as the child points to the last chip; and "This is the four chips" as the child points to the last chip. Other recent evidence, using a trick game methodology to decrease overestimates and underestimates of children's knowledge (Frye et al., 1989), also indicated that some 3- and 4-year-olds use a last-word rule rather than connecting counting to cardinality, because they answered assertions of the interviewer, "I think there are *x*," by agreeing only with last-word responses even when the interviewer made a counting error and the child noticed and identified the counting as incorrect.

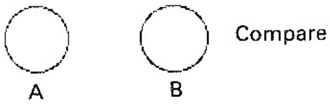
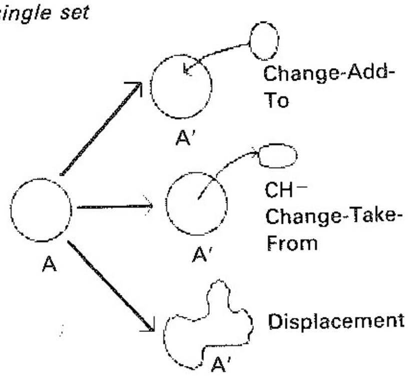
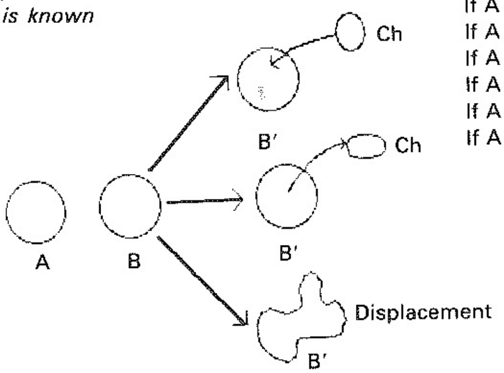
Some children may immediately relate the counting meaning to a cardinality meaning, and many 4-year-olds come to make such a count-to-cardinal transition in word meaning, in which the word shifts from the count reference (a reference to the last counted object) to a cardinality reference (a reference to the numerosity of all of the counted objects). Such a transition requires the child to gather conceptually all the counted entities so that the cardinality reference can be to all of these entities. This conceptual gathering together is called *cardinal integration,* following a related use of *integration* by Steffe et al. (1983). This count-to-cardinal transition allows children to begin to understand numerical equivalence, and to add and subtract in certain situations. (These are discussed in the following sections.) The reverse cardinal-to-count transition in word meaning is required in order for a child to be able to count out a given number of objects. When instructed, "Get five cars," or "Give me five dolls," the child must shift from the cardinal meaning of the *five* to a counting meaning of *five* and know that when counting, the counting must stop when *five* is said. The child must also be able to remember the goal of counting (*five*) while carrying out the counting and to monitor the counting in order to stop at the required word. It may take some time for children to be able to present their counting activity to themselves sufficiently to anticipate this cardinal-to-count transition. Of 28 children, 2 and 3 years old who were last-word responders (children who gave the last counted word in response to a how-many question), 22 did not show any cardinal-to-count transitional ability on any trial. After being told that a row of objects had x of those objects, they were not able to predict what number they would say when they counted the last object in the row. These children did not even guess x; most of them tried to count to find out what they would say last.

There is still controversy concerning the relationship between children's counting behavior and their understandings of counting. I have argued that this relationship is more complex than a simple "skills-first" or "understanding-first" approach (Fuson, 1988, Chapter 10) and discuss evidence concerning the developmental relationships among the Gelman and Gallistel (1978) how-to-count principles. There are many aspects of accurate counting, and children may understand that some of these are required for correct counting before they are able consistently to meet these requirements; they also may carry out accurately certain aspects of counting before they understand that these are required for the counting to be correct. Relationships among a child's saying an accurate sequence of number words, carrying out correct correspondences between number words and objects, and knowing at least a last-word rule vary with the number of objects. For small sets, most children do the first two before the third. For sets between about 4 and 7, children produce correct sequences but may produce correct correspondences and no last-word rule, or may use a last-word rule but make incorrect correspondences. For sets between about 7 and 16, these aspects are ordered: sequence, last-word rule, then correct correspondences. For even larger sets, children may use a last-word rule while producing neither correct sequences nor correct correspondences. With respect to developing understandings of relationships between counting and cardinality, certain aspects of these relationships may be understood earlier for small sets than for large ones.

EQUIVALENCE AND ORDER RELATIONS ON SPECIFIED AND UNSPECIFIED NUMEROSITIES

Between any two sets of entities (two unspecified numerosities) or between any two specified numerosities, A and B, one of three possible cardinal relations will be true: The sets or numerosities will be equivalent, A will be greater than B, or A will be less than B. Table 6.1 outlines different relational situations that have been studied. Piaget's conservation of numerical equivalence situation stimulated many studies of the second and especially of the third type of situation: understanding the effect of a transformation on the relationship between two sets of entities. Various strategies for determining the relation between two unspecified numerosities or two specified numerosities are outlined in the right-hand column

TABLE 6.1
Equivalence and Order Relations on Specified and Unspecified Numerosities

Situations	*Strategies*
Comparing two static sets  A ≅ B or A > B or A < B?	*Unspecified numerosity strategies* General perceptual Length Density Matching: move objects Matching: not move objects If there is no extra, A ≅ B If there is extra, A ≠ B If extra is in A, A > B If extra is in B, B > A
Comparing the pretransformation state to the postransformation state of a single set  A ≅ A′ or A > A′ or A < A′	*Specified numerosity strategies* Subitize Subitize sums Count If $N_A = N_B$, then A ≅ B If $N_A \neq N_B$, then A ≠ B If $N_A < N_B$, then A < B If $N_A > N_B$, then A > B *Nature of the transformation* If Change-Add-To, then A′ > A If Change-Take-From, then A′ < A If Displacement, then A′ ≅ A
Comparing the postransformation state of one set to an untransformed set whose relation to the pretransformation state of the first set is known  A ≅ B′ or A > B′ or A < B′ depends on the transformation and on the relation of A to B	*Nature of the transformation and the original relation* If A ≅ B, then the relationship of A to B depends on the transformation as above If A > B and Ch^+, then A ? B′ If A > B and Ch^-, then A > B′ If A > B and Displ, then A > B′ If A < B and Ch^+, then A < B′ If A < B and Ch^-, then A ? B′ If A < B and Displ, then A < B′

of Table 6.1. Both the unspecified numerosity strategies and the specified numerosity strategies can be used whenever objects are present, whether or not the numerosities of those sets are initially specified. The basis for making the relational judgment is similar for all unspecified numerosity strategies: They depend on identifying and locating any extra objects. The basis for making the relational judgment for the specified numerosity strategies is also similar: Knowledge about the relations on the obtained specific number words is required to determine the relation on the objects. The transformational strategies are only used in situations where a transformation is made; in such situations, the unspecified and specified numerosity strategies can also be used. Empirical strategies (matching and counting) are, however, the only reliable strategies to use in the static comparing situations, and in the Change-Add-To and Change-Take-From situations resulting in A? B′ in the table.

Young children learn important aspects of these comparison strategies before they start school, and learning about these strategies continues in the early years of school. There is a huge literature on children's learning in this area, much of it a reaction to Piaget's original book on number (Piaget & Szeminska, 1941). In spite of the many studies, however, our picture of how children's understanding develops in these different comparison situations is still unclear in several places. A summary outline of this development is presented in Table 6.2. This summary and the brief discussion that is possible here of course ignore many subtleties in this learning. A much fuller discussion is available in Fuson (1988, chap. 8).

At least by age 3, children can use perceptual strategies for comparing two sets, and they understand that adding things to a set means that the set has more and that taking some away means that the set has less. The perceptual strategies, including the use of length or density when objects are arranged in rows, continue to be very powerful during much of the preschool years. During this time children will attend to a transformation on a set and ignore the relation that existed on that and another set originally, but as children learn the cultural tools of counting and matching, these tools become strategies that are increasingly trusted and used in comparison situations. When children learn these tools, of course, depends upon their own culture, so the age may vary widely. The ages given in Table 6.2 primarily reflect research done with English-speaking children in the United States and England.

Children can correctly carry out counting or matching and know how to use the counting or matching information to make an equivalence judgment before they will choose to carry out these strategies voluntarily. If asked to count or match, although they can do so, they will choose to use perceptual strategies instead. In situations in which information obtained from perceptual strategies conflicts with that obtained from the quantitative strategies of counting or matching, children initially will use the information from perceptual strategies. In these cases they may say things like, "This seven has more than that seven does," thus saying the specific numerosities obtained by counting but choosing to use the

TABLE 6.2
Development of Strategies for Establishing Equivalence and Order Relations on Specified and Unspecified Numerosities

Age	Perceptual	Quantitative	Transformation and Original Relation	Transformation Alone
3	General perceptual Length + density Choose L more than D		Like transformation alone, cannot use both transformation information and information about the original relation	Ch+ means changed set has more Ch− means changed set has less Displacement changes the quantity only if it looks like there are more after the transformation
4	General perceptual L + D, L, D	Ch Count, Match UI		
4½	General perceptual L + D, L, D	Ch UI Count, Match		
5½	General perceptual L + D, L, D	Ch, Count, Match UI		
6		Ch, Count, Match UI →	For Ch+ and Ch− begin to consider transformation and original relation and compare size of the difference in each case using count or match For Displacement, relate count and match results to the Displacement transformation Inductive generalization that Displacements do not affect the equivalence relation	Deduction of invariance of displacements even with perceptually misleading information (identity conservation plus transitivity) Immediate equivalence judgment with justification Irrelevance of transformation No addition or subtraction
7			Deduce conservation from reversible counting and matching: reversibility or from simultaneous length and density: compensation	
7½	Estimation L × D (development continues)	Cardinal Number: Seriated,embedded number-word sequence		

Note: Ch is choose to use that strategy, UI is being able to use information if it is provided or if the experimenter suggests that the child obtain it.

perceptual information of "more" (i.e., extra) in one row rather than using the numerosity information that the last words are the same. Eventually, if children are asked to count or match, they *will* use this counting or matching information to make the equivalence judgment even though it contradicts the perceptual information. They may do this first for equivalent sets and then later for unequal sets, where they have to use an order relation on the cardinal numbers to decide which has more. Finally, they will carry out counting or matching voluntarily and will use this information rather than the misleading perceptual information. There are wide individual differences in when these changes are made, with a few 3-year-olds and more 4-year-olds voluntarily counting or matching, and some 5-year-olds still not accepting count or match information over perceptual information when these conflict.

This conflict between perceptual and quantitative strategies occurs in all of the comparing situations outlined in Table 6.1, and the same developmental path through this conflict seems to occur in static and transformation situations. The urge toward counting is so strong in many 5- and 6-year-olds that they try to count even when objects are hidden. I once did a Bruner version of the Piagetian conservation of numerical equivalence task in which one set was hidden while the transformation was made and the set was kept hidden afterwards, but many children still tried to count the hidden objects. The eventual strength of the counting strategy, especially, has been partially obscured by the many Piagetian conservation studies in the United States in which children have been prevented from counting. Sufficient evidence has accumulated concerning counting, and also concerning young children's competence in matching (see Fuson, 1988; Kwon, 1989), to indicate that the developmental sequence of conflict between perceptual and quantitative strategies summarized in Table 6.2 is fairly robust.

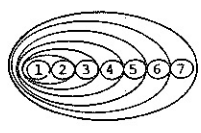
The original Piagetian account of transformation situations, although underemphasizing the roles of counting—and perhaps matching—in dethroning the perceptual strategies, is accurate, in that children do not stop their thinking about these situations even when they can carry out counting or matching accurately. Something seems to impel them on to understand the nature of the transformation itself. Thus, initially children may have to count or match when faced with a transformed situation (and may even use perceptual strategies if prevented from counting or matching); because they must obtain the equivalence information empirically by counting or matching, they are, of course, not conservers. Eventually these children do not count or match when faced with a displaced set: They know that displacement does not affect the original quantity and thus does not affect the original equivalence or nonequivalence relation. This knowledge may be constructed by making an inductive inference from past experiences of counting or matching sets (and be justified by statements that the transformation does not change the quantity or that nothing was added or subtracted); or it may be a deductive inference based either on conceptual knowledge that enables the objects to be transferred mentally to the pre-transformation state and tracked during this reversal (and justified by statements about reversibility) or on knowledge that enables the child to consider length and density simultaneously (and justified by statements about compensation of length and density). Counting or matching may play a role in these inductive or deductive inferences. In Fuson, Secada, and Hall (1983) children spontaneously counted or matched and also gave one of the Piagetian justifications on the very same trial. Sometimes the counting or matching occurred first and the justification seemed to be an attempt to explain the empirical result, and sometimes the justification was given first and then the child counted or matched as if not certain that the explanation was really correct. At present adequate data do not really exist to address the hypothesized developments at ages 6 and 7 that are given in the transformation columns in Table 6.2. We need research in which children are allowed to count or match if they want but are also asked to explain their answer. Studies also need to report the different justifications separately to ascertain if there is any developmental order in them, and children's understanding of all of the justifications needs to be ascertained rather than having questioning stop with the child's first choice of a justification.

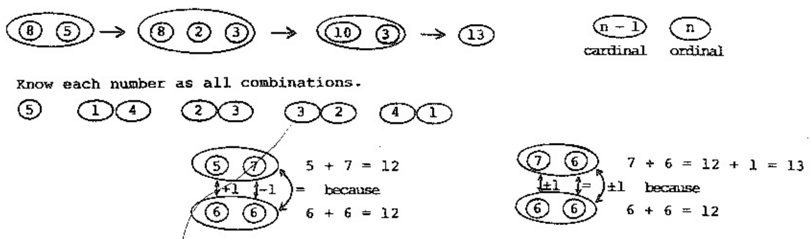
ADDITION AND SUBTRACTION

We know at this time very little about relationships between the conceptual structures children construct and use for addition and subtraction and those they construct for equivalence and order relations, but addition and subtraction and equivalence and order relational situations are clearly related. The Change-Add-To and Change-Take-From transformations of a single set are two very fundamental addition and subtraction situations. When these transformations are carried out on one set in an original relation to another set, a Compare situation (a standard subtraction situation) is related to a second Compare situation; such situations are integer addition and subtraction situations (see, for example, Vergnaud, 1982). Thus, the mathematical situations first studied by Piaget form a rich complex of mathematical situations whose comprehension by children we still only partially understand.

The initial relationships children construct among sequence, counting, and cardinal meanings of number words have already been discussed. Children continue to construct increasingly complex relationships among these meanings that enable them to solve addition and subtraction situations in increasingly sophisticated and efficient ways. In the United States children move through a developmental sequence of such constructions that in most cases are little affected by classroom instruction and frequently are even carried out in the face of active opposition by teachers who may forbid counting or the use of fingers in the classroom. The relationships that are established change the very nature of the sequence of number words: The sequence moves from being a rote, meaningless series of utterances to being constituted by number words that come to stand for objects in the counting procedure and that can take on cardinal meaning as the sum of the words earlier in the sequence. Various levels of meaning that the number word sequence goes through are outlined in Table 6.3. The first four lines in the table (through the Unbreakable List level) have already been discussed. Different parts of the sequence may be developing at different levels

TABLE 6.3
Developmental levels within the number-word sequence

Sequence Level	*Meanings Related*	*Conceptual Structure Within the Sequence and Relationships Among Different Number-Word Meanings*	
String	Sequence	onetwothreefourfivesixseven	Words may not be differentiated.
Unbreakable List	Sequence	one-two-three-four-five-six-seven-	Words are differentiated.
	Sequence-Count	one-two-three-four-five-six-seven	Words are paired with objects.
	Sequence-Count-Cardinal	one-two-three-four-five-six-seven→[seven]	Counting objects has a cardinal result.
Breakable Chain	Sequence-Count-Cardinal	[four]→four-five-six-seven →[seven]	The addends are embedded within the sum count; the embedded first addend count is abbreviated via a cardinal-to-count transition in word meaning.
Numerable Chain	Sequence-Count-Cardinal	four five six seven; [four]; PI; keeping-track method for the second addend	The sequence words become cardinal entities; a correspondence is made between the embedded second addend and some other presentation of the second addend.
Bidirectional Chain/Truly Numerical Counting	Sequence-Count-Cardinal ↕ Sequence-Count-Cardinal		The sequence becomes a unitized seriated embedded numerical sequence; both addends exist outside of and equivalent to the sum; relationships between two different addend/addend/sum structures can be established; addends can be partioned.



Note: A rectangle drawn around related meanings indicates meanings that have become integrated. A number word alone has a sequence or count meaning; a number word enclosed by a bracket has a cardinal meaning.

simultaneously: A child may start counting from given words less than ten while still learning words just before twenty.

At the Breakable Chain level children become able to begin counting at any point in the sequence. The sequence and counting meanings become merged (signified by the rectangle around these meanings at this level), children become able to consider objects that present an addend as also at the same time presenting the sum (one addend becomes embedded within the sum count), and they can move from the cardinal meaning of a given addend ("There are three cats") to the sequence/counting meaning of that addend as the last word said in counting the objects for that addend without needing actually to count the objects or even needing the objects to be visible. They then continue the count of the second addend objects as at the earlier level and make a final count-to-cardinal transition to find the total number of objects. The second addend objects must also be able to be embedded within the sum objects if such children are to continue the count from the first addend. In Secada, Fuson, and Hall (1983) we found many first graders who initially did not embed the second addend within the sum and answered that when counting all the objects they would count the first object in the second addend as *one* or as *five* (or whatever the second addend actually was).

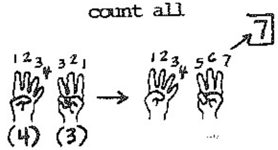
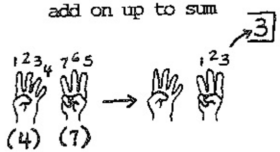
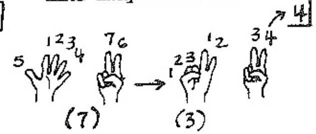
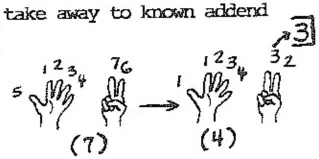
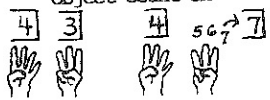
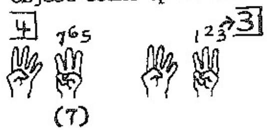
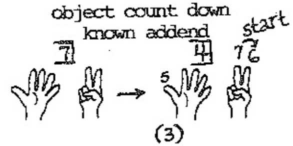
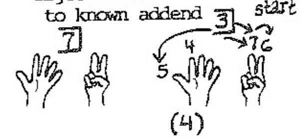
At the next Numerable Chain level all three meanings—sequence, count, and cardinal—are merged, and the sequence words themselves become the objects that present the addends and the sum in addition and subtraction situations. No objects are used to present either addend; the number words are said beginning with the first addend word (or the addend taken by the child to be the first addend) and as many words are then said as are in the second addend (for 6 + 8, start with *eight* and say six more words: *nine, ten, eleven, twelve, thirteen, fourteen*). When the second addend is very large, it is necessary to use some method of keeping track of how many more words are said. Children do this by matching the words said to some known set of entities (e.g., a pattern of six fingers that are extended successively as each word is said or an auditory pattern of speaking three words and then three more words) or by counting the words said (*nine* is one, *ten* is two, *eleven* is three, . . . , *fourteen* is six").

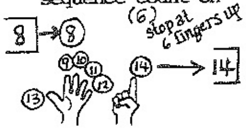
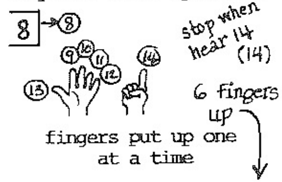
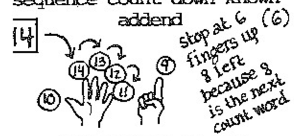
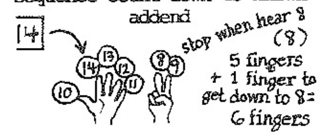
Finally, at the highest level the sequence is a unitized, seriated, embedded, bidirectional, cardinalized sequence. There has been much less research about this level than about the lower levels, and the several conceptual structures portrayed at this level in Table 6.3 may, in fact, occur at different times; that is, this level may eventually be differentiated into different levels. At this level children can construct relationships between two different addend-addend-sum situations and can chunk addends into parts for more convenient adding or subtracting. These various addition and subtraction strategies have been called *derived fact* or *thinking strategies* in the research literature. They usually have not been differentiated, and deciding their developmental sequence is complicated (as is the research on the other sequence levels) by the fact that children can use pattern presentations of numbers to solve problems with small numbers before they can do so in general (e.g., they can count on 1 or 2 before counting on in general and may be able to move one object from addend to addend before moving two objects).

The actual counting that children do in addition and subtraction situations at each level is pictured in Table 6.4. The first two columns show forward counting procedures (one to find a sum and one to find a missing addend), and the second two columns show backward counting procedures. The first and third column procedures reverse each other, and the second and fourth column procedures reverse each other. At the second and third levels, the counting for the first and third columns is governed by the objects for the second addend (one counts forward or backward the number of objects in the second addend), whereas the counting for the second and fourth columns is governed by the auditory count (up to the sum word for the second column and down to the known addend word for the fourth column) and the number of words counted is then found. For most children, counting down is considerably more difficult than counting forward, so there are more errors in the backward counting procedures. The sequence–counting–cardinal relationships are also somewhat more difficult to establish in the backward procedures, introducing other kinds of errors. There are two different backward procedures, and children sometimes confuse them, ending with one object too many or one object too few. The second counting down procedure shown in Table 6.4 may be conceptually based (saying *eight* may mean "eight and one taken away"), or it may be only a procedure not related to the underlying addend structure ("I say four words backwards, and the last word I say is the answer"). There is also a discontinuity between the object counting procedures at Level 1 for the backward procedures but not for the forward procedures, because all of the object counting at Level 1 is forward counting.

The drawings in Table 6.4 show situations in which a problem is given in number words or in written numerals and children then have to count out objects to make the two known quantities. In textbooks and in research studies addition and subtraction situations are also given with objects already presenting the quantities; in such cases the earlier steps would be omitted. Objects are shown for Level 1, but children frequently count out fingers as objects also. At Level 2, children may initially count out objects for the second addend or they may use fingers; this finger use may involve a known pattern of fingers that is made or recognized rather than counted. At Level 3, fingers are not sets of objects that are counted on or counted back as part of the sum, but instead constitute a keeping-track procedure matched to the number words used to count on or count back the sum. The number words themselves present the addends embedded within the sum (i.e., they present the quantities of the addends and the sum), while the fingers (or other keeping-track procedures) just match the number words and then either stop the counting when the correct second addend finger pattern has been produced (in the first and third columns) or present the second addend for

TABLE 6.4
Developmental levels of addition and subtraction solution procedures

Level	*Addend + Addend = [Sum]*	*Addend + [Addend] = Sum*	*Sum − Addend = [Addend]*	*Sum − [Addend] = Addend*
LI	count all  4 + 3 → 7	add on up to sum  4 + ? = 7 ? = 3	take away known addend  7 − 3 = 4	take away to known addend  7 − ? = 4 → ? = 3
LII Fst Add Abb	object count on  4 + 3 → 7	object count up to sum  4 + ? = 7 ? = 3	object count down known addend  7 − 3 = 4	object count down to known addend  7 − ? = 4 → ? = 3
LII Sec Add RT	sequence count all (1)(2)(3)(4) (5)(6)(7) → [7] hear 3 words (3) 4 + 3 → 7	sequence count all up to sum (1)(2)(3)(4) (5)(6)(7) stop at 7 (7) → [3] hear 3 words 4 + ? = 7 → ? = 3	sequence count down known addend and then down to one (7)(6)(5) (4)(3)(2)(1) hear 3 words (3) [4] 7 − 3 = 4	sequence count down to known addend and then down to one (7)(6)(5) (4)(3)(2)(1) hear 3 words before hear (4) (4) 7 − ? = 4 → ? = 3

L III	sequence count on  8] → ⑧ (6) stop at 6 fingers up ⑭ → 14] fingers put up one at a time 8 + 6 ⟶ 14	sequence count up to sum  8] → ⑧ stop when hear 14 (14) 6 fingers up fingers put up one at a time 8 + ? = 14 ⟶ ? = 6	sequence count down known addend  14] stop at 6 (6) fingers up 8 left because 8 is the next count word fingers put up one at a time 14 − 6 ⟶ 8	sequence count down to known addend  14] stop when hear 8 (8) 5 fingers + 1 finger to get down to 8 = 6 fingers fingers put up one at a time 14 − ? = 8 ⟶ ? = 6
L IV	forward doubles plus one ⑦ + ⑥ = ⑫ + ① = ⑬ ⑥+⑥ 7 + 6 = 13	forward doubles plus one +1 (⑦ + ? = ⑬) +1 ?=6 ⑥ + ⑥ = ⑫ 7 + ? = 13 ? = 6	backward doubles plus one +1 (⑬ − ⑦) +1 = ⑥ ⑫ − ⑥ 13 − 7 = 6	backward doubles plus one +1 (⑬ − ?) +1 = ⑥ ⑫ − ⑥ = ⑥ 13 − ? = 6 ? = 7
	forward up-over-ten 5] ⑧ + ⑤ = ⑧ + ② + ③ = ⑩ + ③ ⑩ 8 + 5 = 13	forward up-over-ten 5] ⑧ + ② + ③ to make ⑬ ⑩ 8 + ? = 13 ? = 5	backward down-X-over-ten ⑤ ⑬ − ⑤ = ⑬ − ③ − ② = ⑩ − ② ⑩ = ⑧ 13 − 5 = 8	backward down-over-ten to X ⑬ down to ⑩ = 3 ⑩ down to ⑧ = 2 5] 13 − ? = 8 ? = 5

Note: A number in a right-hand bracket, 4], means a cardinal number (a number that tells how many); a circled number is a sequence number (a number within the counting sequence); and a number in parentheses, (4), means that this number is monitored in a keeping-track process so that some count can end at that number or after that many numbers have been said.

the answer after the counting stops at the sum or the known first addend word (the second and fourth columns). Finally, at Level 4, derived fact strategies are carried out either by operating on the cardinalized sequence and moving through the sequence in chunks or by moving unitized number words from addend to addend within the sequence.

Between Levels 1 and 3, children make major advances with respect to both addends. The counting of the first addend becomes abbreviated as children embed the first addend within the sum and move from counting all to counting on. Some kind of keeping-track process for the second addend enables children to use sequence solution procedures instead of just counting objects. Because different researchers using different tasks have reported on each of these kinds of advances, we do not yet have definitive evidence concerning the developmental relationships between these two advances. Therefore, Level 2 could be considered to have two sublevels, each reflecting an advance for one addend (see Fuson, in press-a). Or both such sublevels could be viewed as transitional, and only three major levels could be identified (Fuson, in press-b). The developmental progression outlined in Table 6.4 also is not the only possible progression. The structure of the number–word sequence and the way in which fingers present addition and subtraction both seem to affect the developmental sequence of solution procedures used by children in a given culture. Other developmental paths are discussed in Fuson and Kwon (this volume, chapter 15).

The addition and subtraction strategies described in Table 6.4 are not rote procedures. They all require a considerable amount of conceptual understanding (see Fuson, 1988, Chapter 8, for a discussion of the conceptual structures involved). Counting at each level requires a conceptual operation of constructing unit items for that level, moving from perceptual unit items at Level 1, in which each object is taken as an identical countable object, to Level 2, where unit items are simultaneously in an addend and in a sum, to Level 3, where the words are considered by a child as the unit items, to Level 4, where the unit items are even more abstract and are able to be separated and combined outside of their addend structure while also staying within it (see Steffe & Cobb, 1988, and Steffe et al., 1983, for more discussion of various unit items children use in addition and subtraction situations). Conceptual uniting operations, *cardinal integrations*, are also required at each level in order to form the unit items into addends and into the sum; the results of these cardinal integrations provide the reference for the cardinal meaning of a number word.

There is a considerable literature in English concerning a range of addition and subtraction situations, particularly addition and subtraction word problems. The main categories of such word problems (these are actually possible real-world addition and subtraction situations) are shown in Fig. 6.2. Each situation involves three quantities, any one of which can be unknown, yielding a large number of kinds of addition and subtraction problems. The performance of children in the United States on a range of these problems has been summarized

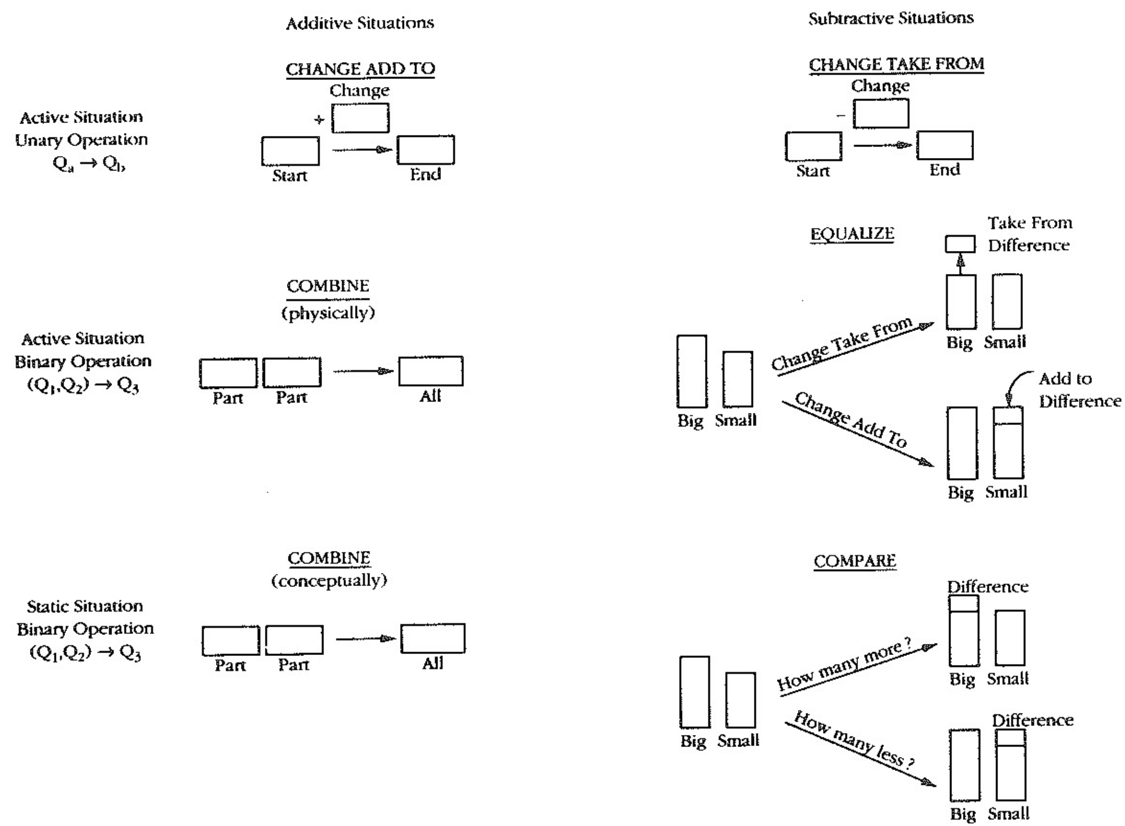


FIG. 6.2. Real world addition and subtraction situations.

in several papers (e.g., Carpenter & Moser, 1983; Fuson, 1988, chapter 8; in press-a, in press-b; Riley, Greeno, & Heller, 1983). Initially children directly model the actions in the addition or subtraction situation, using the strategies described in Table 6.4 that reflect the problem situation. For example, they add on or count up for a Change-Add-To problem in which the change is unknown, but they take away or count down for a Change-Take-From problem in which the end quantity is unknown. At this direct modeling level some types of problems cannot be solved by many children. Later, solution procedures may be more freed from the given situation and are no longer direct models of the problem situation. For example, some children choose to solve all subtraction problems by counting up even for situations that fit the third or fourth columns in Table 6.4.

The present mathematics curriculum in the United States (manifested chiefly through textbooks, because there is no national curriculum) mostly ignores how children think about numerical situations (see Fuson, in press-b, for a review). Children in the United States are provided with a very restricted sample of addition and subtraction situations (Stigler, Fuson, Ham, & Kim, 1986) compared to children in the Soviet Union, who solve problems from the whole range of possibilities. The problems used in the United States are generally only the simplest kinds of problems, which many children can already solve when they begin kindergarten. Textbooks in the United States initially present pictures of objects to count for addition and subtraction problems, but suddenly expect children to solve numeral problems without objects. Flashcards and drills are used to help children memorize the facts, and counting may even be forbidden, being viewed as immature and as interfering with later competence. In fact, we found that when children were given opportunities in the classroom to move through the usual developmental sequence in Table 6.4 up to Level 3, counting on for addition and counting up for subtraction, these counting procedures were efficient and comprehensible enough to be used in multidigit addition and subtraction of up to 10 places if the children used one-handed finger patterns to keep track with their nonwriting hand (Fuson, 1986a, 1986b; Fuson & Briars, 1990; Fuson & Secada, 1986; Fuson & Willis, 1988). Learning to subtract by counting up did not interfere with children's understanding of Change-Take-From situations and made subtraction as easy as addition, because the keeping-track process used the sum rather than a known finger pattern to tell a child when to stop counting. First graders of all ability levels learned these Level 3 procedures for all sums and differences to 18 (i.e., through $9 + 9 = 18$ and $18 - 9 = 9$), a considerable acceleration of the usual expectations for children in the United States, who may not even be given the opportunity to add and subtract all sums to 18 in the first grade (Fuson et al., 1988). Children were quite accurate and fast at such counting and easily used counting on and counting up to find addition and subtraction combinations they did not know when adding and subtracting multidigit numbers in second grade.

There are a number of factors that complicate the effort to understand the conceptual structures for addition and subtraction that are possessed by a given child. Children solve a particular problem in a range of ways and may not use their most advanced solution procedure. Subitizing or subitizing plus adding enable children to carry out procedures with certain small numbers before they can do so in general. Memorized facts enable certain problem types to be solved that cannot be solved for unknown facts. Conceptual structures can be affected by instruction and previous experiences. However, it is clear that children can construct conceptual structures that enable them to understand and solve many different kinds of addition and subtraction problems and that mathematics curricula in the United States considerably underestimate the kinds of problems children can successfully engage.

In first or second grade, children begin to construct multiunit conceptual structures for multidigit numbers. In all of the conceptual structures and solution procedures discussed so far, each number is a unitary collection of single unit items. Larger numbers require children to construct multiunits of ten, hundred, and thousand, from which these larger numbers are composed. These multiunits enable children to understand and use English number words and the standard base-ten written marks for four-digit numbers. Research concerning children's understanding of these larger numbers and of multidigit addition and subtraction is summarized in Fuson (1990), and multiunit conceptual structures are discussed there. Instructional issues concerning multiunit numbers are discussed in Fuson (in press-b), and disadvantages of English words for multiunit numbers are described in Fuson and Kwon (this volume, Chapter 15).

CONCLUSION

For several years young children need to present numerical situations to themselves in some concrete way using objects and, later, number words as objects. The originally separate sequence, counting, and cardinal meanings of number words become related and finally integrated over this period so that the number-word sequence itself becomes the primary conceptual tool for solving addition and subtraction situations. This sequence eventually becomes an embedded, seriated, cardinalized, unitized, numerical sequence.