math expressions

Number and Operations in Base Ten (NBT) in Math Expressions

BUILDING A NEW STANDARD OF SUCCESS

Heinemann

Dr. Karen Fuson, Program Author of ***Math Expressions*** and Professor Emerita of Learning Sciences, School of Education and Social Policy, Northwestern University

THE NUMBER AND OPERATIONS IN BASE TEN (NBT) STANDARDS

The NBT standards focus on understanding place value symbols and number words for whole numbers and decimals and how these symbols and words are used in adding, subtracting, multiplying, dividing, and comparing multi-digit numbers and decimals. The learning progression moves from teen numbers in Kindergarten; to 100 in Grade 1; to 1,000 in Grades 2 and 3; to 1,000,000 in Grade 4; and then to meanings of and operations on decimals in Grades 5 and 6.

This progression begins with an emphasis on understanding and then moves in the next grade to an emphasis on fluency. Initially students are to use concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. They relate the strategy to a written method and explain the reasoning used. For multi-digit multiplication and division, students illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. Students initially invent methods but soon see accessible, research-based, mathematically desirable variations of the standard algorithms that are standard algorithms themselves.

Math Expressions approaches multi-digit computation in above ways. The special MathBoards provide supports for students to learn to make math drawings to show multi-unit quantities. Students draw hundreds, tens, and ones in Grades 1, 2, and 3, and thousands and ten-thousands in Grade 4.

Math Expressions uses array/area models in Grade 3 for single-digit multiplication and division. These are extended and generalized to multi-digit multiplication and division in Grades 4 and 5. *Math Expressions* uses Secret Code Cards to show the meaning of place values for whole numbers and for decimals. These cards and the math drawings help students explain their methods using concepts of place value, as specified in the CCSS and as always done in the *Math Expressions* Math Talk Community (for more information see the author paper *Math Talk Community*).

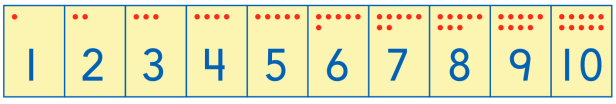
The grade-placement of topics within *Math Expressions* also has always been and is now still consistent with the CCSS placement of topics. There is a major focus in Kindergarten on teen numbers as composed of ten ones and some more ones. Extensive place value work in Grade 1 extends to adding within 100 with regrouping. In Grade 2, adding and subtracting within 1,000 is a major focus, providing extensive experience as a basis for fluency within 1,000 in Grade 3. In Grade 4, place value is extended to 1,000,000; adding and subtracting are generalized to these larger numbers using the standard algorithm and are brought to fluency. The single-digit multiplication and division array and area models from Grade 3 are extended to multi-digit numbers and are related to methods of multi-digit multiplying and dividing by a single-digit number and multiplying 2-digit times 2-digit numbers. Decimal fractions are related to fractions and are compared. In Grade 5, multi-digit multiplication moves to fluency, and division is extended to 2-digit divisors. Extensive work with all four operations with decimals is a major focus. In Grade 6, performing operations with decimals moves to fluency.

MATH EXPRESSIONS CONCEPTUAL SUPPORTS BY GRADE LEVEL

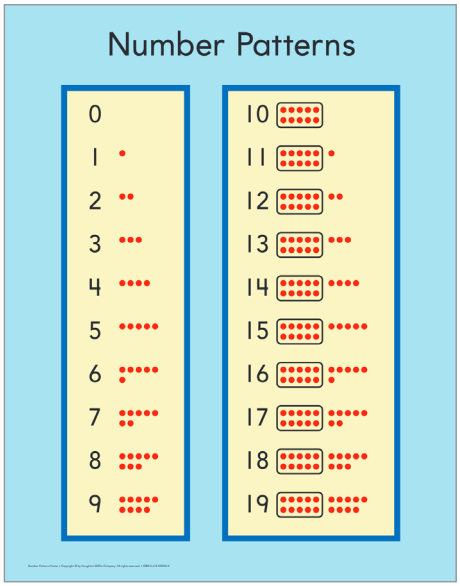
Kindergarten

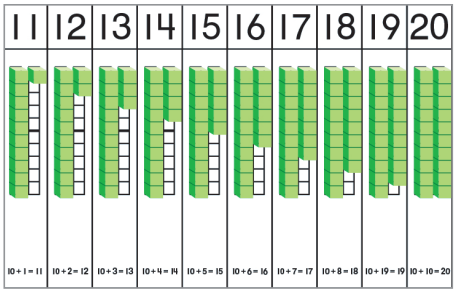
Students in all grades see and make numbers in groups of ten with subgroups of five used for numbers 6 through 10. A Number Parade and Giant Number Cards that show the 5-groups help children learn to visualize the quantities from 1 through 10. Children relate the visual 5-groups to the five groups in their fingers, and they make numbers one to 10 with their fingers. Children can use these visual models when adding and subtracting. In later grades, they draw ones and tens using 5-groups for multi-digit adding and subtracting.

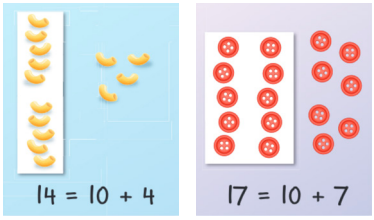
Kindergarten children then focus on teen numbers and do many activities to see a teen number as ten ones and some more ones. They see and discuss the structure of numbers from 1 to 20 in the teen Number Patterns chart. They count out objects to make a teen number and group ten of the objects to see the ten ones inside the teen number. They make drawings of such ten ones and the loose ones. Foam Number Tiles in which the single digits fit on the 0 in the 10 tile help children visualize the 0 hiding under the single digit to see that a teen number like 18 is 10 + 8. These tiles show their numbers on the back with dots in patterns. Children use ten sticks that show ten ones and unit cubes to show numbers 11 to 20 in order on an 11 to 20 Board, and they see and say an equation (e.g., 15 = 10 + 5) for each teen number.

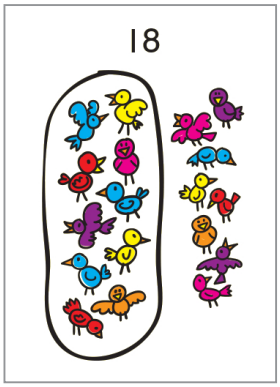


Number Parade

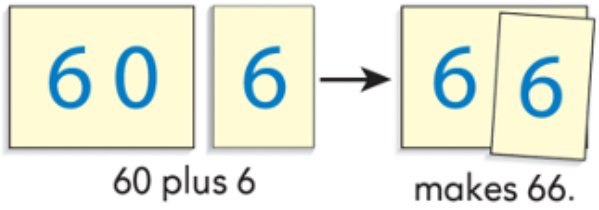


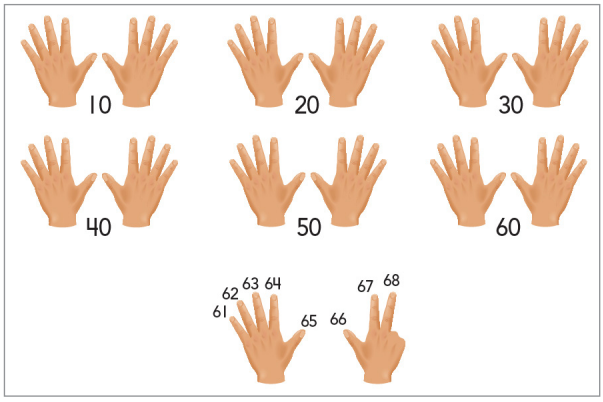


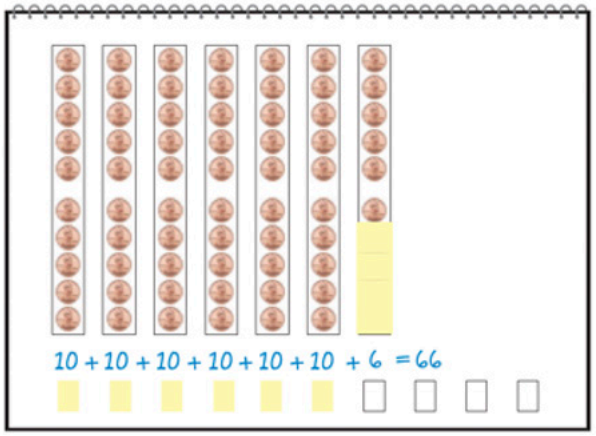


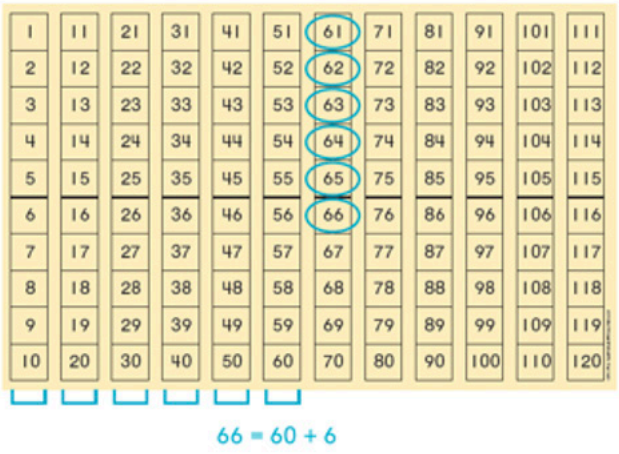


Kindergarten children build throughout the year their understandings of the count to 100 and the groups of 10 involved in that count and in the English number words. They count columns of 10 pennies by tens and then count on single pennies as shown on the Money Flip Chart below. The ten penny strips have a dime on the back of each strip and children later count by tens for the dimes. Children also do such counts by tens and by ones on the 120 Board that shows numerals from one to 120 in vertical columns of 10. Children flash 10 fingers for each count by 10 and then flash single fingers to show the quantities involved in those counts and in the numerals involved. The vertical arrangement allows them to see how the numbers in the tens place are the same in each decade and the ones increase by one



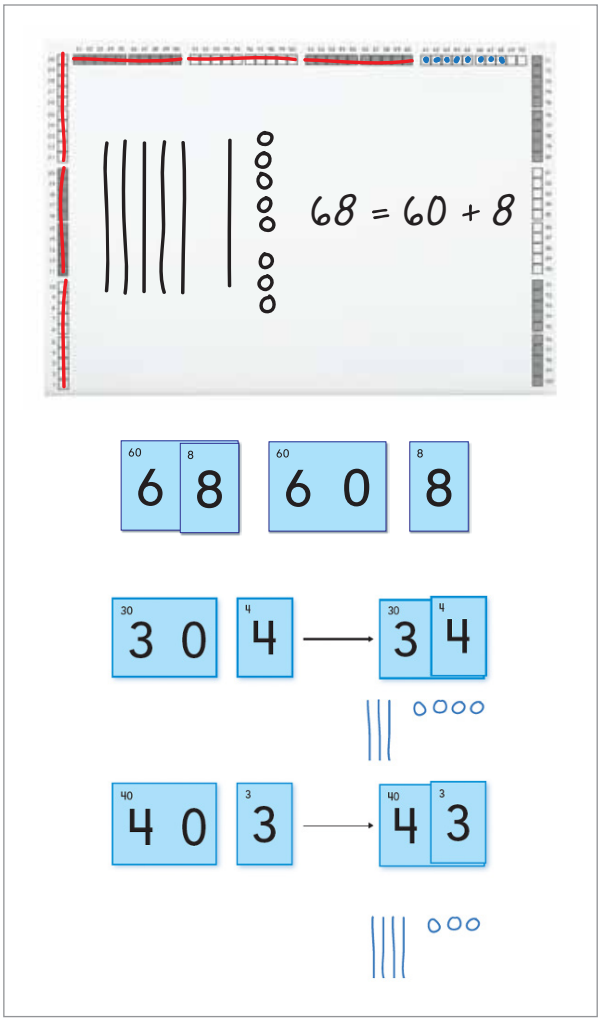




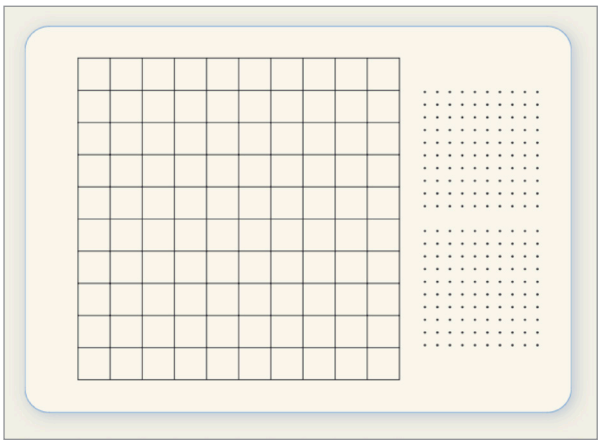


Grade 1

Grade 1 children do intensive work throughout the year understanding place value, written numerals, and English number words. They do routines with the Money Flip Chart and the 120 Board shown above for kindergarten. They use regular tens words like *six tens eight ones* as well as the English words *sixty eight*. They also use a MathBoard with a Number Path to see numbers to 100 grouped by tens with two 5-groups in each group of ten. They use two 10-by-10-centimeter dot grids on the back of the Mathboard to draw ten-sticks on a column of ten dots and so build up numbers to 100 as tens and ones. They make a box around one 10-by-10 grid to see 100 as ten tens and as 100 ones. They then draw any number within 100 as tens and ones by using quick-tens (vertical sticks) and ones (dots or circles).



Secret Code Cards extend the Kindergarten work with teen numbers to show any 2-digit number as a decade and ones (e.g., 68 = 60 + 8), where the 8 can fit on the 0 in the 60 to show 68, but children can visualize the 0 hiding under the ones digit to maintain place-value numbers as tens and ones quantities. Children add numbers within 100 with regrouping using math drawings of the numbers to show tens and ones. They relate the math drawing to a written method and explain their reasoning, making a new ten in the drawing and in the written problem when needed. At all grades word problems are continually interwoven with work on multi-digit computation to give meaning to the operations.



The drawings of tens and ones and the Secret Code cards help children understand and explain comparisons of 2-digit numbers. They can see that four tens are more than three tens and it does not matter how many ones there are

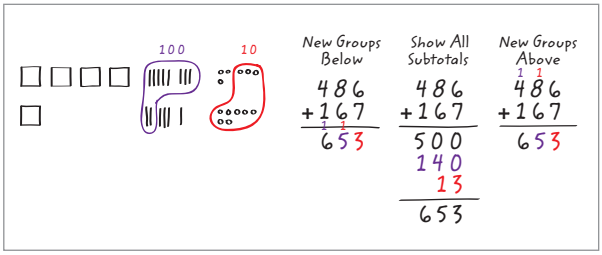
Grade 2

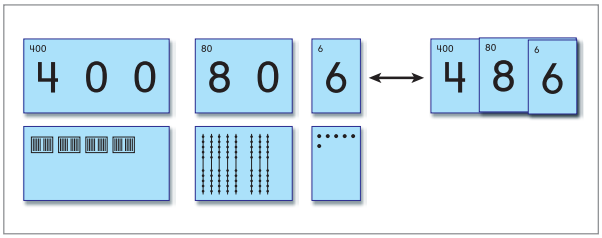
Secret Code Cards and math drawings of hundreds, tens, and ones are extended to 200 and then to 1,000 to enable students to relate quantities of hundreds, tens, and ones to written number symbols and to number words. Math drawings are used in adding and subtracting within 200 to create extended opportunities to understand, explain, and master regrouping with tens and with ones. Then such adding and subtracting with math drawings for sense-making is extended within 1,000. This is an easy extension because only small numbers of hundreds are involved. Many children become fluent within 1,000 with a research-based, accessible, and mathematically desirable way to write the standard addition and subtraction algorithms so that Grade 3 can concentrate on the vital OA focus on concepts and fluency with multiplication and division.

Students invent methods of adding and subtracting, but they also use and explain research-based accessible variations of the standard algorithm that generalize to larger numbers. The first two methods shown appear in the context of a grocery store as Mrs. Green's and Mr. Green's methods. The Show All Subtotals method shows the place values and can be done from the left or from the right (many children prefer to operate from the left to be consistent with reading). The New Groups Below method has several advantages over the more difficult third variation New Groups Above.

- You can see the 13 ones and 15 tens.
- You write these teen numbers in the order in which you usually write them: the 1 ten and then the ones.
- You add the bigger numbers that are there and then add the 1. This is easier and you don't forget the 1.
- You don't change the problem by writing some new number up in the problem space.

In multi-digit subtraction, a very common error is for students to subtract the smaller from the larger number in a column even when the smaller number is on top: 82 – 56 = 34. This error is easy to make in the variation of the standard algorithm in which one alternates ungrouping and subtracting steps. Students in Math Expressions see, use, and explain Mrs. Green's method in which she ungroups everywhere needed first, and then subtracts in all places. With this variation of the standard algorithm, students can ungroup and subtract from the left or from the right. These variations stimulate interesting math discussions as they are compared and explained by students.





Fuson and Beckmann (2012/2013), Fuson and Li (2014), Fuson (2020), and Fuson, Kiebler, and Decker (2024) identified criteria for choosing which algorithms should be emphasized in the classroom. These criteria emphasize ease of understanding and explaining, mathematical accuracy, and place-value attributes important for students to discuss and explain. They also emphasized that there is no one standard algorithm, but rather there are variations in how one writes standard algorithms. The methods emphasized in *Math Expressions* are all such standard algorithms.

Grade 3

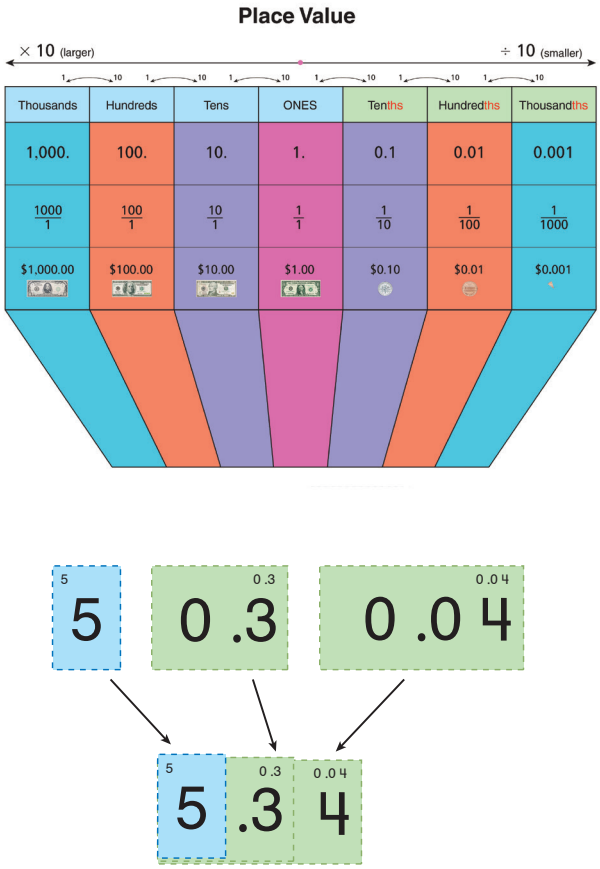
Many students enter Grade 3 no longer needing the math drawing to understand the method they are using and are fluent adding and subtracting within 1,000. Other students take the step of no longer making the math drawing but visualizing the multi-unit quantities as needed or if an error arises. They work on fluency in practice spaced throughout the year.

Grade 4

Students use a place-value poster with money values and symmetrically colored columns to understand relationships between adjacent places in both directions and thus extend multi-digit place-value meanings to decimals. They use decimal Secret Code Cards that show tenths and hundredths as decimals on the front and as money on the back.

Grade 4 students use the 50-by-100 dots on the Grade 4 MathBoard to make place-value drawings through 10,000 as needed, and they use Secret Code Cards to see values of numbers within 1,000,000. They extend the research-based, accessible, and mathematically desirable variations of ways to write the standard algorithms to add and subtract within 1,000,000, explain their methods, and reach fluency with adding and subtracting in this range.

The MathBoard of 50-by-100 dots allows students to make detailed rectangular array and area model drawings for multi-digit multiplication and relate parts of the drawings to parts of the numerical methods in their explanation. Drawings and written methods for multiplication and division are on the next page. After the detailed MathBoard drawings, students move to making area sketches, and they relate parts in these to parts in variations of written multiplication methods as they explain their reasoning. The Expanded Notation Method can help students eliminate errors such as aligning on the left or forgetting what they are multiplying by what. The steps shown in blue can be dropped when they are no longer needed by a student.



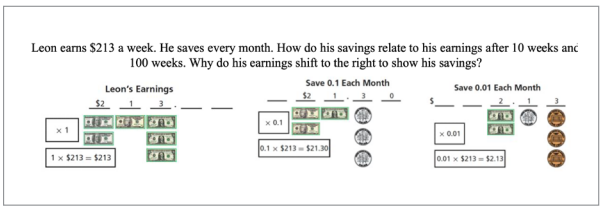
The Place Value Sections accessible standard algorithm uses the area model to organize the multidigit multiplications for students who find it difficult to do so in the Expanded Notation layout. The Common method has a misleading step in the tens units times ones units product (here 60 x 3) which is written in the wrong place (see the 1 from 180 at the top is in the tens place instead of in the hundreds place where it belongs).

Multi-digit division uses array and area drawings to carry out and explain division as finding the unknown factor along the top of the drawing. Students see and explain the place values involved in an expanded notation method, and they also explain the digit-by-digit method where no place values are written.

Grade 5

Grade 5 students extend the Grade 4 array/area diagrams and related ways to write division to 2-digit divisors. Students move to fluency with multiplication, keeping methods meaningful by thinking about array/area diagrams as needed but practicing for fluency without diagrams. Students use the Grade 4 place-value poster shown above and new Secret Code Cards that extend to thousandths to use decimal Secret Code Cards that show all values of places as decimals on the front and as money on the back. They add, subtract, multiply, and divide with these supports explicitly relating each operation with decimals to the related operation with whole-numbers. By reasoning about effects of multiplying or dividing by 0.1 and 0.01, they understand multiplying and dividing by a decimal less than one as having results opposite to those with a decimal more than one: numbers become smaller rather than larger when multiplying and they become larger instead of smaller when dividing. Using such reasoning with any numbers in the tenths and hundredths places enables students to identify patterns in placing the decimal point for multiplication and division with whole numbers and decimals.

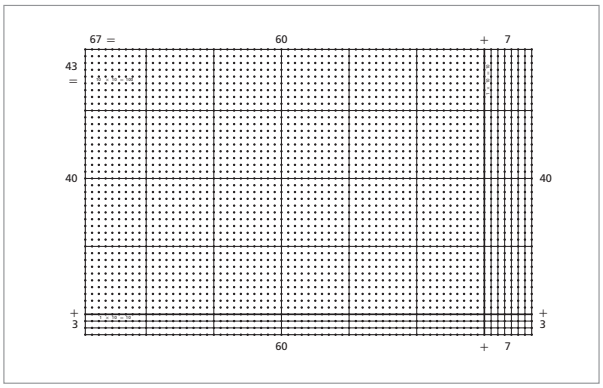
The whole number shifts to the right (gets smaller) when multiplying it by a decimal number because you are taking one-tenth of the numbers in each place. Division by a decimal shifts the dividend to the left (it becomes larger) because one is finding how many tenths are in each number in each place in the dividend.

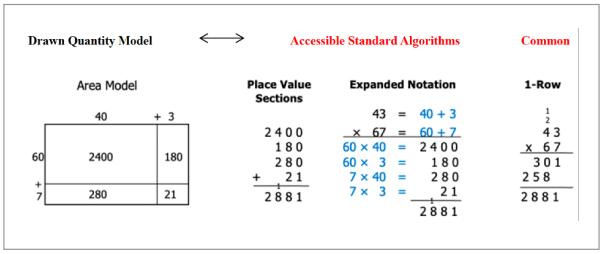


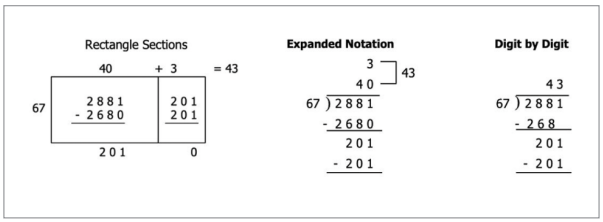
Grade 6

Students build on their Grade 5 conceptual work with dividing by 2-digit divisors and all operations with decimals to become fluent in these operations with the standard algorithms.

Students discuss how operations with numbers in decimal form are like and different from operations with number in fraction form. This enables students to increase their understanding of decimal and fraction notation and meanings.







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